Complementary Coffee Cups

Thomas Banchoff

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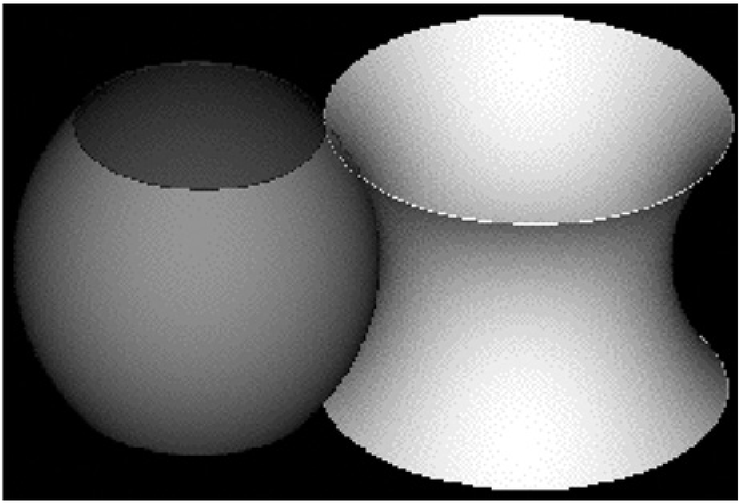
On the very day that I was to begin a unit on volumes of revolution in my beginning calculus class, it was my turn to make the coffee at home. I chose two cups, a convex one from Ireland and a concave one made in Scotland, both surfaces of revolution if you ignore the handles. I noticed that their profile curves fit together, convenient for carrying upstairs. Feeling gallant, I wanted to offer the cup holding more coffee to my wife, but which one was that? It just wasn't clear by inspection which one had the greater volume. At that moment, we believe that a new calculus problem was born.

I washed out the cups and took them to class. I told the story and asked the students to vote on which one held the greater volume. There was no consensus.

One student suggested approximating the profile of the cup by a parabola and using a formula for the volume of revolution. Before we could try that, another student suggested filling the cups with water. Since I had brought some water with me, that is what we did.

Surprisingly, when we poured the water from one full cup to the other and back again, there was no appreciable loss of fluid. For practical purposes, the cups had the same volume.



In the class, we had a theoretical purpose, namely to investigate properties of volumes of revolution, and now we had a good problem to work on. Given a profile curve for one cup of revolution about an axis, how far away should a second axis be so that the volume of this “complementary cup” would be the same as that of the first one? It was clear that the volume would be too small if the second axis were too close to the first, and also that we could make the second cup have arbitrarily large volume by moving the axis far enough away. What distance would be just right?

We phrased the condition for equal volume in terms of our formulas for volumes of revolution. Since it is traditionally easier to work with volumes of revolution about the x-axis, we considered a cup obtained by revolving the graph of the positive function $f(x)$ defined over the interval from a to b. We had established that the desired volume was the integral from a to b of the area of the vertical cross section at x, namely $\pi f(x)^2$.

The complementary cup could then be described by revolving that same curve around a horizontal line $y = k$ for some k. In this case the area of the vertical cross section would be $\pi(k - f(x))^2$. What value of k would guarantee that the volumes of the two cups would be the same?

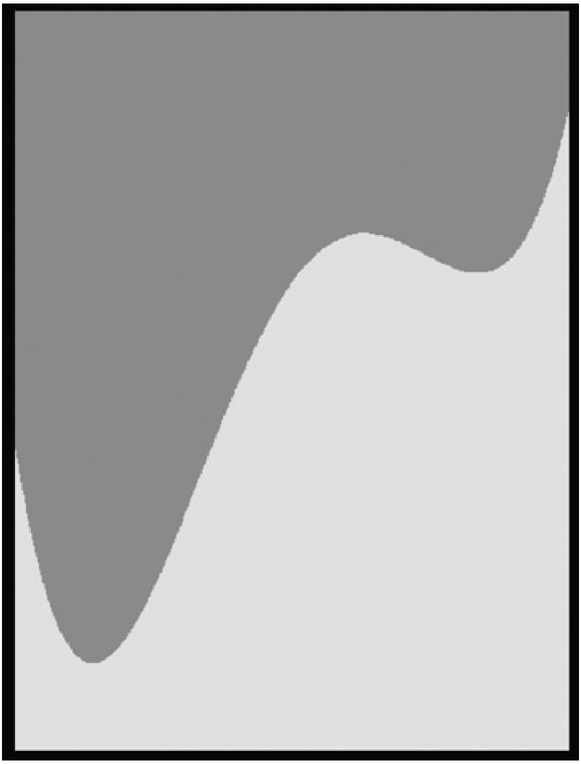
The problem then became easy to solve:

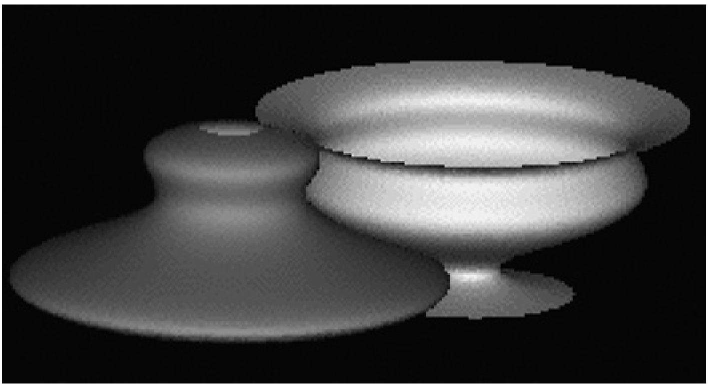
$$\pi \int_a^b (f(x))^2\,dx = \int_a^b (k - f(x))^2\,dx$$
$$= \pi k^2(b-a) - 2k\pi \int_a^b f(x)\,dx + \pi \int_a^b (f(x))^2\,dx.$$

It follows that either $k = 0$, a trivial solution, or

$$k(b-a) = 2\int_a^b f(x)\,dx.$$

This means that the area of the rectangle of height k is twice the area under the graph of f. In other words, the area below the graph of f has to be the same as the area above the graph. The graph divides the rectangle into two pieces of equal area, a very satisfactory, if somewhat mysterious, conclusion.





When an answer comes out that nicely, there has to be a good geometric reason behind it. I mentioned my classroom experience to my colleague Alan Landman, and he said there must be some connection with the beautiful theorem of Pappus about centers of gravity, and indeed there is. Unfortunately, that topic has all but disappeared from many calculus syllabi, and so students often miss it. It is worth salvaging.

Pappus of Alexandria, writing in the third century of our era, showed that the volume of revolution of a region about an axis is the area of the region multiplied by the distance traveled by the center of gravity, that is, the area times $2\pi d$, where d is the distance of the center of gravity from the axis. In our case, the y-coordinate y_c of the center of gravity (x_c, y_c) of the region under the graph of $y = f(x)$ is given by the formula

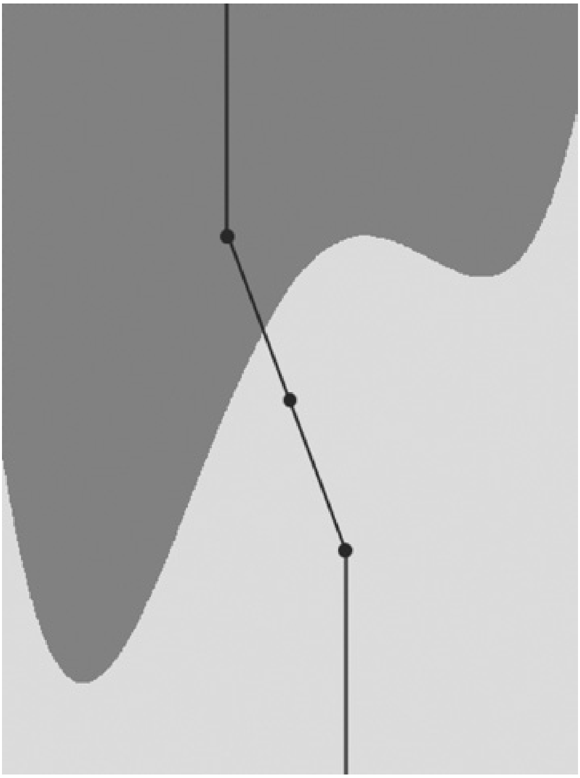
$$y_c = \frac{1}{2}\int_a^b (f(x))^2\,dx \Big/ \int_a^b f(x)\,dx.$$

Thus,

$$2\pi y_c \int_a^b f(x)\,dx = \pi \int_a^b (f(x))^2\,dx,$$

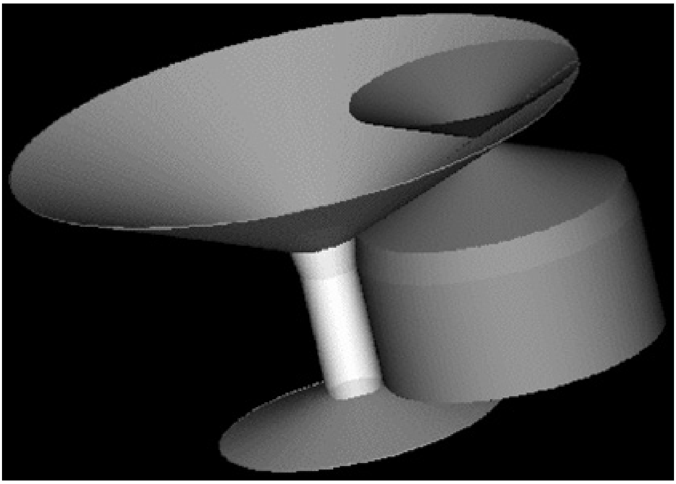
which equals the volume of revolution.

One of the basic properties of centers of gravity is that the center of gravity of the union of two non-overlapping regions of equal area is halfway between the two centers of gravity. Thus, the center of the rectangle is the midpoint of (x_c, y_c) and the center of gravity (x_c', y_c') of the part above the graph. It follows that the distance from y_c to the x-axis is the same as the distance from y_c' to the other horizontal side of the rectangle, so the two volumes of revolution have to be the same.



There is one small condition that has to be met in order for the above calculations to be met. Observe that k is twice the average height of the graph of $y = f(x)$ over the interval. The usual condition for a volume of revolution is that the profile curve should not intersect the axis, so we need to assume that $f(x)$ at each point is less than twice the average value of the function over the interval, something valid for most coffee cups, but not, for example, for a martini glass.

It turns out that the above analysis still works, even if a portion of the region under the curve goes above the line $y = k$. Revolving the graph about this line will produce possibly a number of volumes of revolution over the intervals where the graph is totally above the line or totally below the line. Curiously enough, the shape formed by rotating the profile curve of the martini glass looks very much like the little glasses that bartenders provide for the other half of a double martini. But not for breakfast!



Afterthoughts. When my wife and I visited Korea somewhat after the investigation of the complementary coffee cups, we were introduced to celadon pottery, one example of which is a matched pair of vases with the same profile, but revolved around two different vertical axes. They do not have the same volume, but I'll bet the potter could make that happen if we just sent the result proved above.

Is this a new problem after all? We will see, first of all from the letters to the author or the editor, and secondly, if the problem miraculously appears in the next editions of all of the duelling calculus textbooks.

Acknowledgments. Thanks to David Eigen for the computer graphics illustrations, and credit to Brown University/John Forasté for the photograph of the author.

Exchange Rates Between the States

A TABLE

Exhibiting the value of a Dollar in each of the United States; and practical Theorems for exchanging the currency of either, into that of any other.

To exchange from / to	*N. Engl. States & Virginia.*	*Pennsylvani. Jer. Dela & Marylan.*	*New York and N. Carolina.*	*S. Carolina and Georgia.*
* New England States and Virginia	Dollar 6/0	Add one 4th.	Add one 3d.	Subtract $\frac{1}{9}$ twice
Pennsylvania, N. Jersey Delaware and Maryland	Subtract one 5th.	Dollar 7/6	Add one 15th.	$\times$ 3 $\frac{1}{8}$, & $\div$ 5
New York and North Carolina,	Subtract one 4th.	Subtract one 16th.	Dollar 8/0	To $\frac{1}{2}$ add $\frac{1}{6}$ of the $\frac{1}{2}$
South Carolina and Georgia,	Add two 7ths.	Add $\frac{1}{2}$ $\frac{1}{7}$ that $\frac{1}{2}$ & $\frac{1}{2}$ that $\frac{1}{7}$	$\times$ 2 & Sub. $\frac{1}{7}$ Product.	Dollar 4/8

* The New England States are, New Hampshire, Massachusetts, Rhode Island, and Connecticut.

The American Tutor's Assistant Revised; or, A compendious system of Practical Arithmetic; containing the several rules of that useful science concisely defined, methodically arranged, and fully exemplified. The whole particularly adapted to the easy and regular instruction of youth in our American schools. Joseph Crukshank, Philadelphia, 1809, page 110.

Editor's question: Can you spot the arithmetic error in the table?