No Arithmetic Cyclic Quadrilaterals

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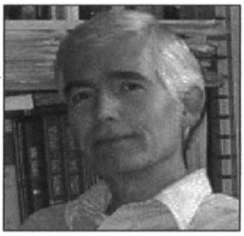

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An arithmetic quadrilateral is one with integer sides in arithmetic progression and integer area. For example, the quadrilateral in Figure 1 is arithmetic and has area 16. This notion generalizes that of arithmetic triangles, which have been and continue to be a fascination. The term *arithmetic triangle* was defined in **[2]**, where the infinite family of all such was determined. There it was shown that an arithmetic triangle with relatively prime sides $x, x+\delta$, and $x+2\delta$ exists if and only if $|\delta|$ is 1 or is a product of primes $p_i \equiv \pm 1$ modulo 12. The proof shows how a primitive solution of the Diophantine equation $g^2 - 3h^2 = \delta^2$ corresponds to a triangle with sides $x = 2g - \delta, 2g, 2g + \delta$. The reader can experiment with the 13-14-15 triangle or the 15-28-41 triangle. (See also the more recent paper by MacDougall **[7]**.) Now every triangle is inscribable in a circle, but the same is not true of quadrilaterals. Quadrilaterals (like the one in Figure 2) that are so inscribable are said to be cyclic. In view of the triangle result, it is remarkable that there are no arithmetic cyclic quadrilaterals. Buchholtz and MacDougall **[1]** proved this in 1999 but they used an elliptic curve argument. Our purpose is to give an elementary proof using classic results of Euler and Fermat.

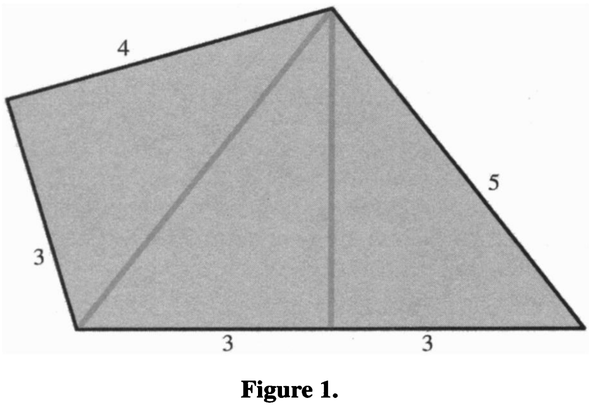

Figure 1.

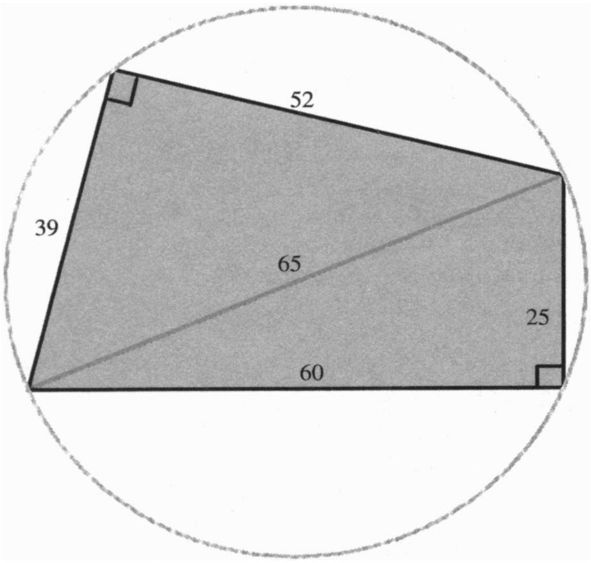

Figure 2.

There is a nice formula for the area of a cyclic quadrilateral in terms of the length of its sides; it is often attributed to Brahmagupta [**4**] and generalizes Heron's area formula for triangles, but it is not nearly as well known. The area A of a cyclic quadrilateral having side lengths a, b, c, d satisfies

$$A^2 = (s-a)(s-b)(s-c)(s-d),$$

where s is the semi-perimeter $(a+b+c+d)/2$. If the sides are integers in arithmetic progression so that

$$a = x, b = x+\delta, c = x+2\delta, d = x+3\delta,$$

then Brahmagupta's equation becomes

$$A^2 = x(x+\delta)(x+2\delta)(x+3\delta). \tag{1}$$

Euler [**6**] proved that the product of four positive integers in arithmetic progression cannot be the square of an integer. Thus, the Buchholtz-MacDougall result follows.

Since a proof of Euler's result is not readily accessible, we will explain why it is true. Earlier, Fermat showed that four integers in arithmetic progression cannot all be squares [**3**, p. 440]. We use this fact (a proof of which is indicated at the end of this note) together with the easily established fact that odd squares are congruent to 1 modulo 8 to argue our case. We assume that A is an integer, and, without loss of generality, that the four sides a, b, c, d do not have a common proper factor; that is, $(x, \delta) = 1$. We proceed to show that any two of the sides do not have a common proper factor, forcing the four sides to be square integers.

First observe that since x and δ are relatively prime, the only possible common prime divisors of two of the sides $a = x$, $b = x+\delta$, $c = x+2\delta$, $d = x+3\delta$ are 2 and 3, and there are only three choices: (i) 2 could divide a and c, (ii) 2 could divide b and d, or (iii) 3 could divide a and d. Let's see how 2 or 3 might divide A^2 in (1).

Case 1. Suppose that A^2 is odd and a multiple of 3. Then 3 divides x and $x + 3\delta$ but not $x + \delta$ and $x + 2\delta$. The latter two are squares congruent to 1 (mod 3); subtraction shows that $\delta \equiv 0 \pmod 3$, contradicting $(x, \delta) = 1$.

Case 2. Suppose that A^2 is even but not a multiple of 3. Thus δ is odd because only two of the sides are even. If x is even, then 2 divides $x + 2\delta$, but it cannot divide $x + \delta$ or $x + 3\delta$. Thus $x + \delta$ and $x + 3\delta$ are both odd squares and so are congruent to 1 (mod 8); subtraction shows that 2δ is a multiple of 8, contradicting the fact that δ is odd. Similarly, if x is odd, then so is $x + 2\delta$, and these are squares congruent to 1 (mod 8), forcing 2δ to be a multiple of 8.

Case 3. Neither Case 1 nor Case 2 occurs. Then 6 must divide A^2 as follows:

$$a = 6u^2, b = v^2, c = 2s^2, d = 3t^2.$$

The equations $2a + d = 3b$ and $a + 2d = 3c$ yield

$$4u^2 + t^2 = v^2, u^2 + t^2 = s^2, \tag{2}$$

respectively. Since t is odd, the (primitive) Pythagorean triple $(2u, t, v)$ has the form $2u = 2nm$, $t = n^2 - m^2$, $v = n^2 + m^2$, where m and n are relatively prime, $n > m$, and just one of them is odd. Substitution into the second equation in (2) yields

$$n^4 - n^2m^2 + m^4 = s^2. \tag{3}$$

According to Dickson [**3**], Euler proved that (3) has no integer solutions with $|n| > |m|$ except $(n^2,\ m^2) = (1, 0)$. Pocklington gave a nice proof of this using Fermat's method of descent, which is reproduced in [**8**, p. 20] and [**5**]; our conditions on n and m do not allow this choice. Thus the four sides must all be squares, contradicting Fermat's result.

Fermat's impossibility of four squares in arithmetic progression follows from this result too. Fogarty and O'Sullivan [**5**] gave an interesting discussion proving this fact. Here is a direct proof. Suppose $a = u^2, b = v^2, c = s^2, d = t^2$ are in arithmetic progression. Then the equations $a + c = 2b$ and $b + d = 2c$ take the form

$$u^2 + s^2 = 2v^2 \quad \text{and} \quad v^2 + t^2 = 2s^2.$$

Substitution into the equation $u^2t^2 = t^2u^2$ yields

$$u^2(2s^2 - v^2) = t^2(2v^2 - s^2),$$

or

$$2(u^2s^2 - v^2t^2) = u^2v^2 - s^2t^2.$$

Notice that both $uv + st$ and $uv - st$ must be even. Letting $2p = uv + st$, $2q = uv - st$, $n = us$, and $m = vt$, we obtain $nm = p^2 - q^2$ and $(n^2 - m^2) = 2pq$. Squaring the last equation yields

$$n^4 - n^2m^2 + m^4 = (p^2 + q^2)^2,$$

contradicting (3).

As we have seen, arithmetic quadrilaterals do exist (although they are not cyclic). Cyclic quadrilaterals with integer sides of the form $a = x$, $b = x + \delta$, $c = x + 2\delta$, $d = x + 3\delta$ (but noninteger area) abound. Since opposite angles must have the form $\alpha, \pi - \alpha$ and $\beta, \pi - \beta$, one can work with the law of cosines and determine the angles. A computation shows that α and β satisfy

$$\cos(\alpha) = \frac{\delta(3\delta + 2x)}{3\delta^2 + 6\delta x + 2x^2} \quad \text{and} \quad \cos(\beta) = \frac{\delta(3\delta + 2x)}{3\delta^2 + 3\delta x + x^2}.$$

References

1. R. H. Buchhlolz and J. A. MacDougall, Heron quadrilaterals with sides in arithmetic or geometric progression, *Bull. Austral. Math. Soc.*, **59** (1999) 263–269.
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3. L. E. Dickson, *History of the Theory of Numbers*, Vol. II, Chelsea, 1934.
4. H. Eves, *An Introduction to the History of Mathematics*, 5th ed., Saunders, 1983.
5. K. Fogarty and C. O'Sullivan, Arithmetic progressions with three parts in prescribed ratio and a challenge of Fermat, *Math. Mag.*, **77** (2004) 283–292.
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7. J. A. MacDougall, Heron triangles with sides in arithmetic progression, *J. Rec. Math.*, **31** (2003) 189–196.
8. L. J. Mordell, *Diophantine Equations*, Academic Press, 1969.
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Alligation

Alligation is a rule for adjusting the prices and simples of compound quantities.

Case 1 When several simple quantities, and their prices are given, and a mean price of any part of the compound is required.

Rule

As the sum of the several quantities,
Is to their total value;
So is any part of the composition,
To its value.

Examples

1. If 19 bushels of wheat at 6*s* the bushel, 40 of rye at 4*s*, and 12 of barley at 3*s*, be mixed together, what is a bushel of this mixture worth?

B.		*s.*		
19	at	6 = 14		
40	at	4 = 160		
12	at	3 = 36		
—		— *s.* *d.*		
71		)310(4	$4\frac{1}{4}$ *answer.*	

2. A grocer mixed sugars; 2 *Cwt.* at 56*s*. 1 *Cwt.* at 43*s*. and 2 *Cwt.* at 50*s* per *Cwt.* What is 3 *Cwt.* of this mixture worth? *answer* 7*l* 13*s*.

The American Tutor's Assistant Revised; or *A Compendious System of Practical Arithmetic* Printed by Joseph Crukshank, Philadelphia, 1809, page 159.