

NOTES

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Magic “Squares” Indeed!

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1 INTRODUCTION. Behold the remarkable property of the magic square:

$$\begin{bmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{bmatrix}$$

$$618^2 + 753^2 + 294^2 = 816^2 + 357^2 + 492^2 \text{ (rows)}$$

$$672^2 + 159^2 + 834^2 = 276^2 + 951^2 + 438^2 \text{ (columns)}$$

$$654^2 + 132^2 + 879^2 = 456^2 + 231^2 + 978^2 \text{ (diagonals)}$$

$$639^2 + 174^2 + 852^2 = 936^2 + 471^2 + 258^2 \text{ (counter-diagonals)}$$

$$654^2 + 798^2 + 213^2 = 456^2 + 897^2 + 312^2 \text{ (diagonals)}$$

$$693^2 + 714^2 + 258^2 = 396^2 + 417^2 + 852^2 \text{ (counter-diagonals)}.$$

This property was discovered by Dr. Irving Joshua Matrix [3], first published in [5] and more recently in [1]. We prove that this property holds for *every* 3-by-3 magic square, where the rows, columns, diagonals, and counter-diagonals can be read as 3-digit numbers in *any* base. We also describe n -by- n matrices that satisfy this condition, among them all circulant matrices and all symmetrical magic squares. For example, the 5-by-5 magic square in (1) also satisfies the square-palindromic property for every base.

$$\begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix} \tag{1}$$

We must be careful when we read these numbers. The base 10 number represented by the first row of (1) is $17 \cdot 10^4 + 24 \cdot 10^3 + 1 \cdot 10^2 + 8 \cdot 10 + 15 = 194195$. The base 10 number based on the first row's reversal is 158357.

2 SUFFICIENT CONDITIONS. We say that a real matrix is *square-palindromic* if, for every base b , the sum of the squares of its rows, columns, and four sets of diagonals (as in the previous examples) are unchanged when the numbers are read “backwards” in base b . We can express this condition using matrix notation. Let M be an n -by- n matrix. Then the n numbers (in base b) represented by the rows of M are the entries of the vector $M\mathbf{b}$, where $\mathbf{b} = (b^{n-1}, b^{n-2}, \dots, b, 1)^T$, and T denotes the transpose operation. The sum of the squares of these numbers is

$$(M\mathbf{b})^T(M\mathbf{b}) = \mathbf{b}^T(M^T M)\mathbf{b}.$$

Next, the n numbers represented by the rows when read “backwards” are the entries of $MR\mathbf{b}$ where the n -by- n reversal matrix $R = [r_{ij}]$ has $r_{ij} = 1$ if $i + j = n + 1$, and $r_{ij} = 0$ otherwise. Note that $R^T = R^{-1} = R$. The sum of the squares of these numbers is

$$(MR\mathbf{b})^T(MR\mathbf{b}) = \mathbf{b}^T(R(M^T M)R)\mathbf{b}.$$

Hence a sufficient condition for the rows of M to satisfy the square-palindromic property is simply $R(M^T M)R = M^T M$. Matrices A that satisfy $RAR = A$ are called *centro-symmetric* [6]: $a_{ij} = a_{n+1-i, n+1-j}$. Matrices A that satisfy $RAR = A^T$ are called *persymmetric* [4]: $a_{ij} = a_{n+1-j, n+1-i}$. It is easy to see that symmetric matrices that are centro-symmetric must also be persymmetric. Since $M^T M$ is necessarily symmetric, our sufficient condition says that $M^T M$ is centro-symmetric, or equivalently, that

$$M^T M \text{ is persymmetric.}$$

The square-palindromic condition for the columns of M is the square-palindromic condition for the rows of M^T . Hence it suffices to require that

$$MM^T \text{ is persymmetric.}$$

For the first set of *diagonals*, we create a matrix \tilde{M} with the property that each column of \tilde{M} represents a diagonal starting from the first row of M . To do this, we introduce two other special square matrices. Let $P_k = [p_{ij}]$ denote the n -by- n projection matrix whose only non-zero entry is $p_{kk} = 1$. Notice that $P^T = P$, and $P_k M$ preserves the k th row of M but turns all other rows to zeros. Let $S = [s_{ij}]$ denote the n -by- n shift operator where $s_{ij} = 1$ if $i - j \equiv 1 \pmod{n}$, $s_{ij} = 0$ otherwise.

The following properties of S are easily verified: $S^n = I_n$, $S^{-1} = S^T = RSR$, and MS^k shifts the columns of M over “ k steps to the left”. Now define

$$\tilde{M} = \sum_{i=1}^n P_i MS^{i-1}.$$

Hence the i -th diagonal of M , starting from the first row becomes the i -th column of \tilde{M} . By the column condition, these diagonals satisfy the square-palindromic property if the (i, j) entry of $\tilde{M}\tilde{M}^T$ equals its $(n + 1 - j, n + 1 - i)$ entry.

We have

$$\tilde{M}\tilde{M}^T = \sum_{i=1}^n P_i MS^{i-1} \left(\sum_{j=1}^n P_j MS^{j-1} \right)^T = \sum_{i=1}^n \sum_{j=1}^n P_i MS^{i-j} M^T P_j.$$

It follows that $\tilde{M}\tilde{M}^T$ has the same (i, j) entry as $MS^{i-j}M^T$, and the same $(n + 1 - j, n + 1 - i)$ entry as well; if $MS^{i-j}M^T$ is persymmetric, then these entries are equal. Consequently, these diagonals obey the square-palindromic property if

$$MS^k M^T \text{ is persymmetric for } k = 1, \dots, n. \quad (2)$$

Conveniently, (2) also ensures that the counter-diagonals starting from the first row satisfy the square-palindromic property. This can be seen by mimicking the preceding explanation with $\tilde{M} = \sum_{i=1}^n P_i MS^{-(i-1)}$, whereby $\tilde{M}\tilde{M}^T$ has the same (i, j) and $(n + 1 - j, n + 1 - i)$ entry as $MS^{j-i}M^T$. For the other diagonal and

counterdiagonal, we obtain similar results [7], which we summarize in the following theorem:

Theorem 1. *A square matrix M has the square-palindromic property if the following matrices are all persymmetric:*

1. $M^T M$,
2. MM^T ,
3. $MS^k M^T$, for $k = 1, \dots, n$, and
4. $M^T S^k M$, for $k = 1, \dots, n$.

3. SQUARE-PALINDROMIC MATRICES. Next we explore classes of matrices that are square-palindromic. We say that a square matrix A is *centro-skew-symmetric* if $RAR = -A$, that is, $a_{ij} + a_{n+1-i, n+1-j} = 0$.

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 \end{bmatrix} \\ \text{Centro-Symmetric} \end{array} \qquad \begin{array}{c} \begin{bmatrix} a & b & c \\ d & 0 & -d \\ -c & -b & -a \end{bmatrix} \\ \text{Centro-Skew-Symmetric} \end{array}$$

Theorem 2. *Every centro-symmetric or centro-skew-symmetric matrix is square-palindromic.*

Proof: If M is centro-symmetric or centro-skew-symmetric, then the relations $RM = \pm MR$ and $R(S^k)R = S^{-k}$ ensure that M satisfies the conditions of Theorem 1. ■

The theorem is not at all surprising since the collection of rows, columns and diagonals of M read the same backwards and forwards. The next class of matrices, however, satisfies the conditions in a non-obvious way.

We say that A is *circulant* if every entry of each “diagonal” is the same, i.e., $a_{ij} = a_{k\ell}$ if $i - j \equiv k - \ell \pmod n$ or simply $SAS^{-1} = A$. We say that A is *(-1)-circulant* if $SAS = A$.

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix} \\ \text{Circulant} \end{array} \qquad \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix} \\ (-1)\text{-Circulant} \end{array}$$

Notice that the circulant and (-1)-circulant property is preserved under transposing. It is easy to show that the product of two circulant matrices or two (-1)-circulant matrices is circulant, while the product of a circulant and (-1)-circulant matrix is (-1)-circulant. Note that S is circulant, R is (-1)-circulant, and that all circulant matrices are persymmetric since a_{ij} and $a_{n+1-j, n+1-i}$ lie on the same diagonal. Consequently, if M is circulant or (-1)-circulant, the matrices $M^T M$, MM^T , $MS^k M^T$, and $M^T S^k M$ are all circulant, and thus persymmetric. From Theorem 1, it follows that

Theorem 3. *Every circulant or (-1)-circulant matrix is square-palindromic.*

Notice that four of the six square-palindromic identities are not obvious, but two of the diagonal sums are completely trivial!

4. MAGIC AND SEMIMAGIC SQUARES. A *semi-magic square* with magic constant c is a square matrix A in which every row and column adds to c . Using matrix notation, this says that $AJ = cJ = JA$, where J is the matrix of all ones. If the main diagonal and main counter-diagonal also add to c , then the matrix is called a *magic square*. Circulant and (-1) -circulant matrices are always semi-magic, but are not necessarily magic.

A magic square A is *symmetrical* [2] if the sum of each pair of two entries that are opposite with respect to the center is $2c/n$, that is $a_{ij} + a_{n+1-i, n+1-j} = 2c/n$. Notice that a semimagic square with this property is magic.

Like the example below, magic and semi-magic squares do not necessarily satisfy the square-palindromic property.

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Semi-Magic but not square-palindromic

However,

Theorem 4. *Every symmetrical magic square is square-palindromic.*

Proof: The trick is to notice that if M is a symmetrical magic square with magic constant c , then $M = M_0 + cJ/n$, where M_0 is a symmetrical magic square with magic constant 0. But this implies that M_0 is centro-skew-symmetric. Therefore M_0 is square-palindromic and satisfies the conditions of Theorem 1. Thus, since $M_0^T M_0$ and J are persymmetric, it follows that $M^T M = (M_0 + cJ/n)^T (M_0 + cJ/n) = M_0^T M_0 + c^2 J/n$ is also persymmetric. Hence M satisfies condition 1 of Theorem 1. To verify condition 3 (the other cases are similar), notice that

$$MS^k M^T = \left(M_0 + \frac{c}{n} J \right) S^k \left(M_0 + \frac{c}{n} J \right)^T = M_0 S^k M_0^T + \frac{c^2}{n} J$$

is persymmetric for $k = 1, \dots, n$, since M_0 satisfies condition 3 of Theorem 1. ■

Although not all magic squares are square-palindromic, it is easy to see that all 3-by-3 magic squares are symmetrical. Consequently, we have

Theorem 5. *All 3-by-3 magic squares are square-palindromic.*

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An Elementary Proof of Binet's Formula for the Gamma Function

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The present note presents an elementary proof of the following important result of J. P. M. Binet [3, p. 249].

Theorem 1. For $x > 0$ we have

$$\Gamma(x + 1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \cdot e^{\theta(x)} \quad (1)$$

where

$$\theta(x) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-xt} \frac{1}{t} dt.$$

Here Γ denotes the gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Since $\lim_{x \rightarrow \infty} \theta(x) = 0$, from (1) we immediately obtain Stirling's formula

$$n! = \Gamma(n + 1) \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Binet's formula can also be used to prove a more precise version of Stirling's asymptotic expansion

$$\log \frac{n!}{(n/e)^n \sqrt{2\pi n}} = \sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)n^{2j-1}} = \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \dots,$$

where the B_{2j} 's denote the Bernoulli numbers defined by

$$\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} = \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} t^{2j-1}.$$

For, by problem 154 in Part I, Chapter 4 of [2], the inequalities

$$\sum_{j=1}^{2N} \frac{B_{2j}}{(2j)!} t^{2j-1} < \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} < \sum_{j=1}^{2N+1} \frac{B_{2j}}{(2j)!} t^{2j-1}$$