that the maximum of the $O(1/m)$ expressions might not actually be $O(1/m)$. Thus, a few lines later, it is invalid to make the claim

$$\lim_{m\to\infty}\sum_{k=1}^{m}\frac{B_k}{k!}\left[O\left(\frac{1}{m}\right)\right]=\lim_{m\to\infty}O\left(\frac{1}{m}\right)\sum_{k=1}^{m}\frac{B_k}{k!}=0,$$

even though $\sum_{k=1}^{\infty}B_k/k!$ converges.

However, this can be corrected fairly easily. Pick $\epsilon>0$ and find p such that $\sum_{k>p}\frac{|B_k|}{k!}<\epsilon$. Then write

$$\begin{aligned}&\lim_{m\to\infty}\sum_{k=1}^{m}\left[\frac{B_k}{k!}m^{1-k}(m(m-1)\cdots(m-k+2))\right]\\&\quad=\lim_{m\to\infty}\sum_{k=1}^{p}\left[\frac{B_k}{k!}m^{1-k}(m(m-1)\cdots(m-k+2))\right]\\&\qquad+\lim_{m\to\infty}\sum_{k>p}\left[\frac{B_k}{k!}m^{1-k}(m(m-1)\cdots(m-k+2))\right].\end{aligned}$$

The sum in the first term to the right of the equals sign has only p terms in it, and so the method in [**1**] is valid for this term. Thus the first term is within ϵ of $\sum_{k=1}^{\infty}B_k/k!$. Also, it is easy to see that the second term is within ϵ of 0. Equation (3) follows.

REFERENCE

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Integer-Coefficient Polynomials Have Prime-Rich Images

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Since the time of Euler, mathematicians have been investigating polynomials with integer coefficients and the values they take on at integer points. It is well known, for example, that a nonconstant polynomial $f(x)$ with integer coefficients produces at least one composite image [**1**, p. 46].

Math. Mag. **83** (2010) 55–57. doi:10.4169/002557010X480008.

In this note, we use Taylor expansions to improve this elementary result, showing that $f(x)$ takes an infinite number of composite values. Given a positive integer n, we show that $f(x)$ takes an infinite number of values that are divisible by at least n distinct primes, and an infinite number of values that are divisible by p^n for some prime p.

Notation For a nonzero integer c, let $\omega(c)$ denote the number of distinct prime numbers that divide c. For example, $\omega(700) = \omega(2^2 \times 5^2 \times 7) = 3$. Similarly, for an integer c and a prime p, let $\psi_p(c)$ denote the highest power of p that divides c, that is, $\psi_p(c) = e$ if and only if p^e divides c but p^{e+1} does not divide c. For example, $\psi_7(3773) = \psi_7(7^3 \times 11) = 3$.

Now, and for the rest of this paper, let $f(x)$ denote a polynomial with integer coefficients and degree $d > 1$. In this notation, we show that given a positive integer n, there are infinitely many integers b such that $\omega(f(b)) > n$ and, for some prime p, infinitely many integers b such that $\psi_p(f(b)) > n$.

Abundant prime factors Most classic number theory textbooks use the Taylor polynomial to show that $f(x)$ produces at least one composite image; that is, $\omega(f(b)) > 1$ for at least one composite image; that is, either $\omega(f(b)) > 1$ for some integer b or $\phi_p(f(b)) > 1$ for some integer b and some prime p. Following this lead we apply the Taylor polynomial to show that $\omega(f(b)) > n$ for an infinite number of integers b. On route to a contradiction, let us assume that there is a constant N such that $\omega(f(a)) < N$ for all integer a such that $f(a) \neq 0$. Thus, the set of positive integers $\Omega = \{\omega(f(a)) : f(a) \neq 0, a \in \mathbb{Z}\}$ has a largest element, call it $n = \omega(f(b)) > 1$. Suppose that the prime factorization of $f(b) = c$ is given by

$$f(b) = c = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}.$$

Then the dth Taylor polynomial for f based at b is

$$\begin{aligned} f(b + tc^2) &= c + \frac{f^{(1)}(b)}{1!} tc^2 + \frac{f^{(2)}(b)}{2!} t^2 c^4 + \dots + \frac{f^{(d)}(b)}{d!} t^d c^{2d} \\ &= c + c^2 \left(\frac{f^{(1)}(b)}{1!} t + \frac{f^{(2)}(b)}{2!} c^2 t^2 + \dots + \frac{f^{(d)}(b)}{d!} c^{2d-2} t^d \right). \end{aligned}$$

Inspection shows that $f(b + tc^2)$ is divisible by c for every integer t. One also sees, since $f(x)$ is not constant, that the expression in parenthesis is not zero.

From the first of these observations, since we assume that $\omega(f(b + tc^2)) \leq n$, the primes that divide c are exactly those that divide $f(b + tc^2)$, provided $f(b + tc^2) \neq 0$, and the powers of those primes in the factorization of $f(b + tc^2)$ are at least as high as the powers in c. This allows us to write $f(b + tc^2) = \pm p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, where $\alpha_i \leq e_i$ for all $1 \leq i \leq n$, as long as $f(b + tc^2) \neq 0$.

Furthermore, if $\alpha_i < e_i$, then $p_i^{\alpha_i+1}$ divides c^2 and $f(b + tc^2)$ and consequently also divides $c = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, which contradicts the Fundamental Theorem of Arithmetic. Hence, $f(b + tc^2) = \pm c$, if $f(b + tc^2) \neq 0$, which contradicts the Fundamental Theorem of Algebra. Therefore, the set $\Omega = \{\omega(f(a)) : a \in \mathbb{Z}, f(a) \neq 0\}$ is not bounded above and we have proved the following result.

Given an integer $n > 0$, $\omega\big(f(b)\big) > n$ *for an infinite number of integers* b. (1)

High powers of some primes Now assume that $f(x)$ is irreducible. Then $f(x)$ and its derivative $f^{(1)}(x)$ are relatively prime and, therefore,

$$h(x) f(x) + g(x) f^{(1)}(x) = r$$

for some polynomials $h(x)$ and $g(x)$ with integer coefficients, and a nonzero integer r. Then, applying (1), we choose an integer b and a prime p so that p divides $f(b) \neq 0$ but does not divide r. Hence, $h(b)f(b) + g(b)f^{(1)}(b) = r$ and so $f^{(1)}(b)$ is not divisible by p.

Therefore, p and $f^{(1)}(b)$ are relatively prime and so

$$px + f^{(1)}(b)y = 1$$

for some integers x and y. We now write $f(b) = mp^e$ where $\psi_p(f(b)) = e \geq 1$ and work as we did before to obtain

$$\begin{aligned}
&f(b - myp^e)\\
&\quad= mp^e + \frac{f^{(1)}(b)(-myp^e)}{1!} + \frac{f^{(2)}(b)(-myp^e)^2}{2!} + \cdots + \frac{f^d(b)(-myp^e)^d}{d!}\\
&\quad= p^e\left(m\left(1 - f^{(1)}(b)y\right) + \frac{f^{(2)}(b)p^e(-my)^2}{2!} + \cdots + \frac{f^{(d)}(b)p^{e(d-1)}(-my)^d}{d!}\right)\\
&\quad= p^e\left(m(px) + \frac{f^{(2)}(b)p^e(-my)^2}{2!} + \cdots + \frac{f^{(d)}(b)p^{e(d-1)}(-my)^d}{d!}\right)\\
&\quad= p^{e+1}\left(mx + \frac{f^{(2)}(b)p^{e-1}(-my)^2}{2!} + \cdots + \frac{f^{(d)}(b)p^{e(d-1)-1}(-my)^d}{d!}\right).
\end{aligned}$$

Hence,

$$1 \leq \psi_p\big(f(b)\big) = e < e + 1 \leq \psi_p\big(f(b - mp^e)\big).$$

Further, since $f^{(1)}(b - myp^e) = f^{(1)}(b) \neq 0 \pmod p$, the process can be repeated as many times as we please. Therefore, we have proved that the set

$$\big\{\psi_p\big(f(a)\big) : a \in \mathbb{Z},\, p \text{ prime}\big\}$$

is not bounded above. Thus, we have proved the following result.

Given $n > 0$, $\psi_p\big(f(b)\big) > n$ *for a prime* p *and an infinite number of integers* b. (2)

We complete our paper with the following immediate consequences of (1) and (2).

For an infinite number of integers b, $f(b)$ *is not a prime power.* (3)

For an infinite number of integers b, $f(b)$ *is not square-free.* (4)

There are infinitely many prime numbers. (5)

REFERENCE

1. Gareth A. Jones and J. Mary Jones, *Elementary Number Theory*, Springer, London, 1998.

Summary It is well known that a nonconstant polynomial $f(x)$ with integer coefficients produces, for integer values of x, at least one composite image. In this note, we use Taylor expansions to improve this elementary result, showing that $f(x)$ takes an infinite number of composite values. Given a positive integer n, we show that $f(x)$ takes an infinite number of values that are divisible by at least n distinct primes, and an infinite number of values that are divisible by p^n for some prime p.