$$w^* = Q_{i+1}(A)P_1(A)\dots P_i(A)w' = Q_{i+1}(A)0 = 0.$$

COROLLARY (Cayley-Hamilton Theorem). $C_A(A) = 0$.

Proof. It suffices to prove $C_A(A)v$ is the zero vector for every v in R^n. But this is just the Theorem for $i = k$, since v is in $W_k = R^n$ and $C_A(x) = P_1(x)\dots P_k(x)$.

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Names of Functions: The Problems of Trying for Precision

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To students

Are you being confused by a textbook or a teacher who insists that $f(x)$ is the value of a function at the point x, whereas f is the name of the function? I wouldn't be surprised if you were, especially if your text goes on (most of them do) to present formulas like $\frac{d}{dx}\sin x = \cos x$ or $\frac{d}{dx}x^2 = 2x$, apparently without reflecting that, if $\sin x$ is the value of the sine function at x, then $\frac{d}{dx}\sin x$ is, strictly speaking, a meaningless formula. (We differentiate functions, not numbers.) You could get around this particular problem by writing $\frac{d}{dx}\sin = \cos$; but what *are* you to do about the function that has the value x^2 at x? You can't very well write ^{2} as the name of the function. What most people—you (probably), your teacher (quite likely), and your pocket calculator (almost certainly)—want to call the function is x^2, but this would violate the teacher's principles. I want to propose a simple way out of this bind.

First let's recall that there is one kind of function that has a well-established and unambiguous notation already: a sequence. The sequence $\{n^2\}_1^\infty$ is the function whose value at n is n^2, the domain being understood to be the set of positive integers. If you need a different domain you can write symbols like $\{n^2\}_3^\infty$ or $\{n^2\}_{10}^{100}$ or $\{n^2\}_{\text{odd } n}$. The identity sequence is $\{n\}_1^\infty$. In other words, we know a sequence when we meet one because it comes with its domain and its value at each domain point—which is just what every definition of a function is supposed to provide.

Why then don't we denote the function that has the value x^2 at the real number x by $\{x^2\}_R$ or $\{x^2\}_1^\infty$, and so on? The identity is $\{x\}$, and $\frac{d}{dx}\{x\} = \{1\}$, as it should be. This would be unambiguous, compact, and would require no special symbols to be learned.

If you use any notation a great deal you tend to abbreviate it, just as the word "radix" (meaning "root") got cut down to the modern square root sign. If your usual domain for functions is all real numbers, you will probably just write $\{\sin x\}$ instead of $\{\sin x\}_{-\infty < x < \infty}$. After doing that for a while, you will find yourself dropping the braces too—and there you will be with functions named x^2, $\sin x$, and so on, just as the keys on the calculator, or the tables in the textbook say. The difference is that you now ought to be able, on demand, to explain the difference between $\sin x$, meaning a number, and $\sin x$, meaning a function—which is probably what your teacher was hoping for all along.

To teachers

Are you quite, quite sure that when you make students learn that f is a function and $f(x)$ is a value of a function, they are really learning what functions and values are? Or are they just parroting words? I've seen plenty of students who could give you a letter-perfect definition of a derivative but were helpless if you asked them what $\lim_{x\to\pi}\frac{\sin x}{x-\pi}$ is. How many of them can tell you what is the sine of the angle whose sine is x?

Maybe you deplore the calculator people's putting "x^2" on the squaring key. Your most impassioned arguments aren't going to stop them. "If you can't lick 'em, join 'em."

Abbreviations are an important mathematical tool. If we weren't allowed to use them, we'd still be writing $\lim_{y\to x}\frac{f(y)-f(x)}{y-x}$ instead of $f'(x)$. Bourbaki calls this sort of thing "abus de langage;" ordinary people call it shorthand. Admittedly it has disadvantages. The notation $\sin^2 x$ (Gauss is on record as detesting it) is shorthand for $(\sin x)^2$. The notation $\log^2 x$ can mean either $(\log x)^2$ or $\log(\log(x))$. However, people seem to prefer ambiguous notations to cumbersome ones.

In integration theory, a Lebesgue integrable function is not a function at all; it is an equivalence class of functions. Do we indicate this by our notation? Not that I've ever noticed.

I have heard a linguist claim that the eccentric orthography of English serves a useful purpose besides making it possible to have spelling bees: it helps us pick out the correct meanings of words that we see. Perhaps it's just as well that we *don't* use strictly consistent notations.

Paradoxes

Alas, poor Zeno!

Achilles and the tortoise
stomping 'round his bed
and that confounded arrow
whizzing past his head.

"All Cretans are liars."

Epimenides proposed it
to vex his friends (and you!):
If it's true it's clearly false
while if it's false it could be true.

The village barber

Shaves those and only those whose
razors stay on the shelf.
This makes a hairy problem:
does the barber shave himself?

Warning

The pitfalls of semantics
and logic's fickle weather—
are you prepared to cope with
"is not" and "is" together?

—KATHARINE O'BRIEN