A Geometric Interpretation of Knot Complement Gluings

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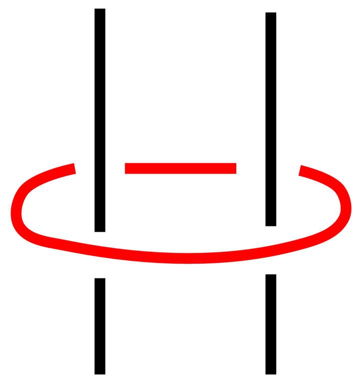
Abstract

We have found exact slope lengths for Dehn fillings yielding two twist 2-bridge knots from the Borromean rings, and found fundamental domains for the universal covers of general 2-bridge knots. These will yield more information about slope lengths.

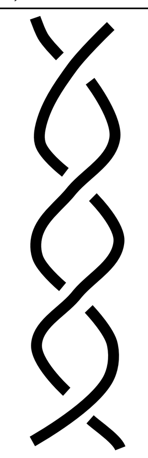
Dehn Fillings and Slope Lengths

A **Dehn filling** on a manifold is gluing a solid torus into a boundary component with gluing information regarding the torus' meridian.

Before Dehn Filling



After (1,2) Dehn Filling Red

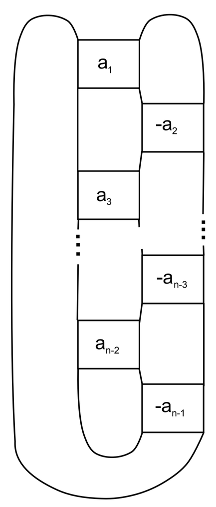


This induces twists on strands passing through the torus. The gluing information is given in terms of **slope lengths** in the universal cover of the manifold in hyperbolic 3-space. This is the length of the curve on the toroidal neighborhood of the boundary component to which the meridian on the torus' surface will be glued.

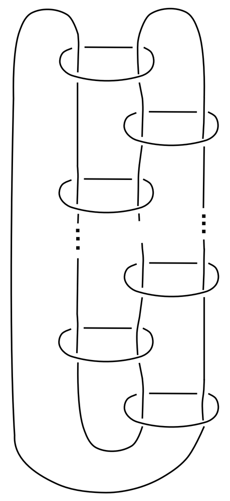
Parent Manifolds

Any $n-1$ tangle 2-bridge knot can be achieved by Dehn fillings on a parent manifold of n Borromean ring complements glued along thrice-punctured spheres.

2-Bridge Knot



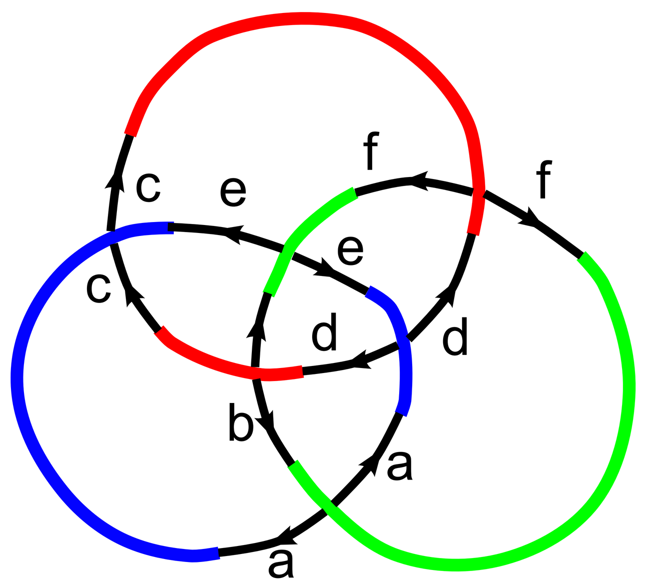
Parent Manifold



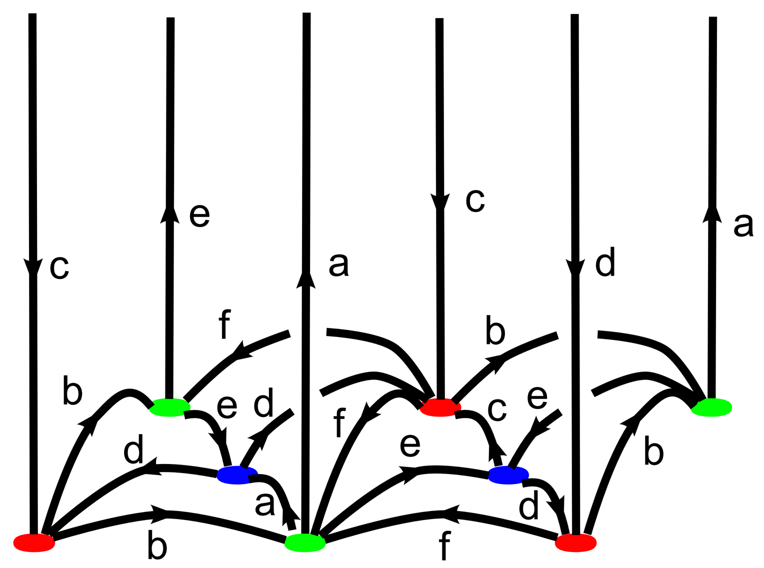
Due to Adams [1], 3-manifolds such as knot complements may be cut and glued along embedded thrice-punctured spheres. These gluings respect the total volume of hyperbolic 3-manifolds additively. This bounds above the volumes of such 2-bridge knots by n times the volume of the Borromean rings' complement. To examine slope lengths for these fillings, we explore the universal cover of the Borromean rings in hyperbolic space, which has a polyhedral fundamental domain.

Fundamental Domain of Borromean Rings' Complement

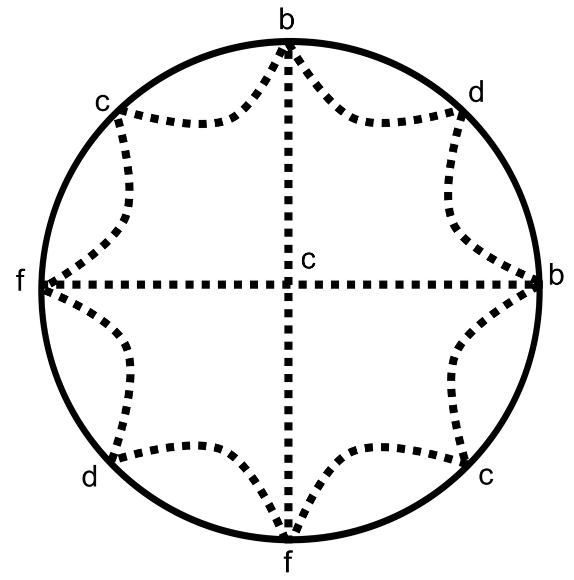
Cell Decomposition

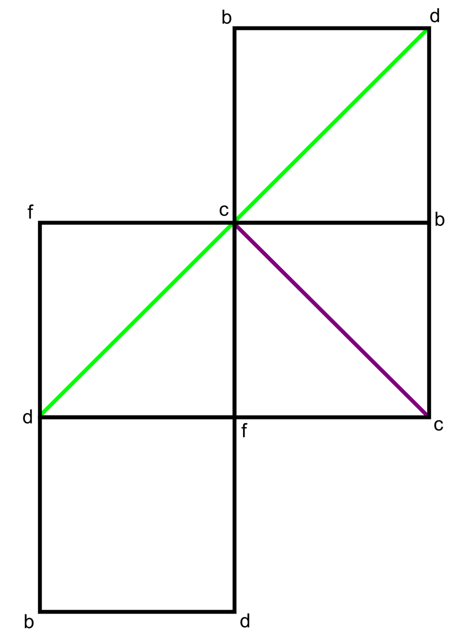


Fundamental Domain



Finding Slope Lengths





Examining the intersections of a horoball including a red point (shown above projected onto the plane with 1-cell intersections) and comparing this neighborhood to a toroidal neighborhood of the red component of the Borrommean rings allows us to develop a fundamental domain for the neighborhood lying on the surface of the horoball.

Using an isometry of the upper half space to center this horoball at infinity, we may use the Euclidean metric on this horoball to measure the length of a meridian and longitude in the fundamental domain, shown in purple and green respectively. The fundamental domain of the torus is made up of unit squares, resulting in the lengths of the meridian and longitude at $\sqrt{2}$ and $2\sqrt{2}$.

This improves Futer, Kalfagianni, and Purcell's lower bound [2] on the volumes of two twist 2-bridge knot complements with at least seven crossings per twist. This bound on $V(M)$, the hyperbolic volume of the knot complement is

$$V(M) \geq \left(1-\left(\frac{2\pi}{l_{min}}\right)^2\right)^{3/2} 2v_8$$

where l_{min} is the minimum slope length of all Dehn fillings and v_8 is the volume of an ideal octohedron, ≈ 3.66866.

Given that both twist regions have at least $m \geq 2$ full twists, the volume of the knots' complement, $V(M)$ will have

$$V(M) \geq \left(1-\left(\frac{2\pi}{\sqrt{2m^2+8}}\right)^2\right)^{3/2} 2v_8$$

General Parent Manifolds for 2-Bridge Links

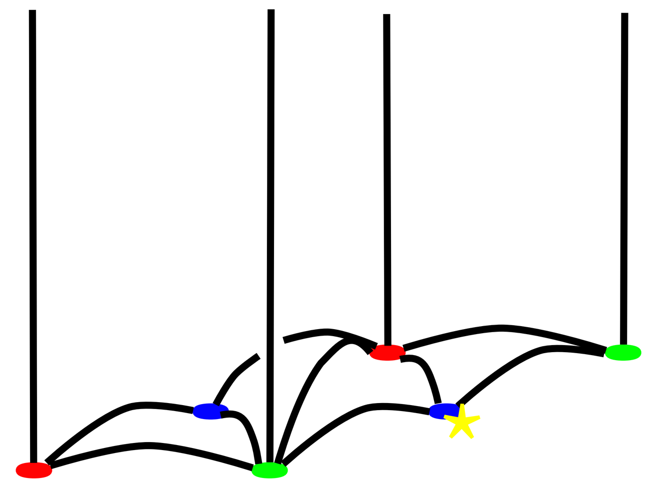
These results are for a very specific class of knots, and in order to generalize to 2-bridge knots with more twist regions, we must understand their universal covers. These are constructed by gluing copies of the Borrommean rings' complement along thrice punctured spheres, which appear as geodesic surfaces in the universal cover. To make this explicit, we reglue the fundamental domain from before to reveal these geodesic surfaces.

We may then glue multiple copies of these together to create a fundamental domain for such a parent manifold. The building blocks and resultant fundamental domain are shown at right. With more work to explore maximal horoballs, more general results about slope lengths can be achieved.

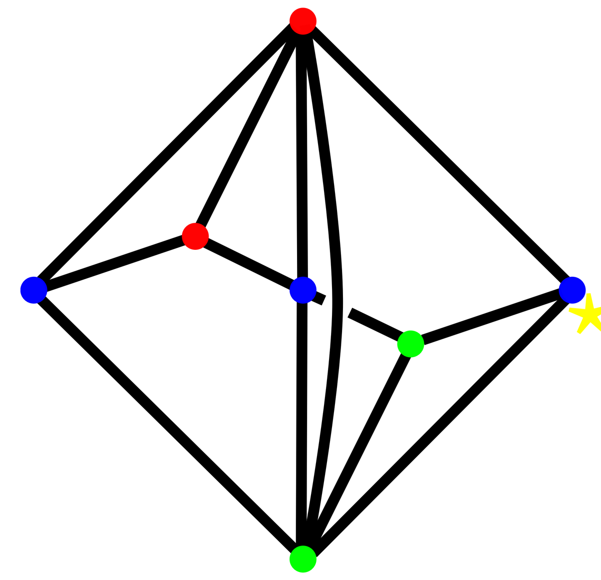
Finding slope lengths for the Borromean ring complements was relatively simple, as the polyhedra in the fundamental domain all included an ideal vertex at the point of infinity. This is not the case after adjusting the fundamental domain to the general case.

Building Block and General Fundamental Domain

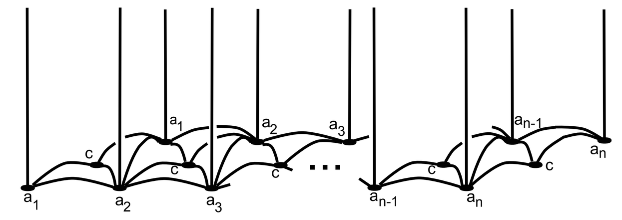
Top Portion of Building Block



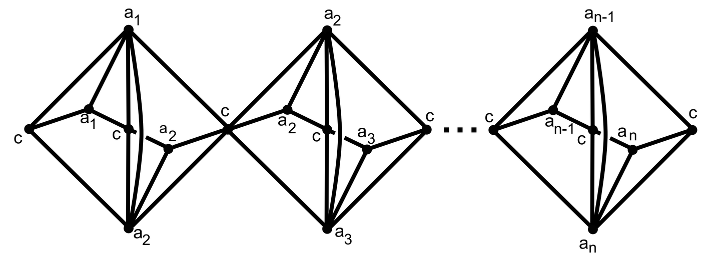
Glued Beneath (seen from above)



Top Portion of General Domain



Glued Beneath (seen from above)



Further Research

1. Given the fundamental domain shown above, how do we find maximal horoballs on all cusps?
2. Can we use this method to understand slopes of hyperbolic Dehn fillings in general?
   - How do these slope lengths affect the change in volume by Dehn filling?
3. Can we use similar techniques on other classes of knots?
   - The Borromean rings offer a very manageable fundamental domain, but other parent manifolds may as well.

Acknowledgements

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References

1. Colin C. Adams,*Thrice-Punctured Spheres In Hyperbolic 3-Manifolds*. Transactions Of The American Mathematical Society, Volume 287, Number 2, February 1985.
2. David Futer, Efstratia Kalfagianni, Jessica S. Purcell, *Dehn Filling, Volume, And The Jones Polynomial*. Journal Of Differential Geometry, 2008.

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