Gergonne’s Card Trick, Positional Notation, and Radix Sort

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The three pile trick  Hold a deck of cards face down and deal 27 cards face up in rows of three, creating three piles each nine high. Overlap the cards in each pile so that your audience can see the values of the cards and so that you can easily pick them up while preserving their order.

Ask a spectator to think of one of the cards, remember it, and tell you which pile it’s in. Announce that you will magically move her card to the middle of the deck.

Pick up the three piles, turning them over so that they are face down, quietly making sure that the pile containing the chosen card is in the middle. Accompany this action by any patter you choose.

Do this twice more, each time putting the chosen pile in the middle. Then count out the deck to the middle card and turn it over, to your audience’s surprise and applause.

I found this classic trick in Rouse Ball [11, p.138] while searching for self-working magic that depends on mathematics rather than dexterity. I have taught it to a fourth grade mathematics club and to precocious first graders. Writing this paper led me to lots of other references, starting with Gardner [4]. You can find several discussions on the internet [2],[10]; Bogomolny [1] provides a Java applet. The trick is named for Joseph Diaz Gergonne (1771–1859), who first published an analysis in Annales de Mathématiques, the journal he founded [5, iv, 1813–1814, pp. 276–284]. Mathematicians regularly return to the problems it raises, sometimes rediscovering or reproving theorems known to previous authors—for example, Dickson in 1895 in the first volume of the Bulletin of the American Mathematical Society [3] and Harrison, Brennan and Gapinski much more recently in Discrete Applied Mathematics [6]. The treatment here explains the trick as a special case of the radix sorting algorithm from computer science.

Base three arithmetic  Most discussions of the trick go on to describe a generalization that clearly depends on base three arithmetic.

Ask the spectator where she wants you to make her chosen card appear in the deck. Tell her that for your magic to succeed she must start counting at 0, not at 1—so the first card is the 0th and the last is the 26th.

Expand the chosen position as a three digit number in base 3. Read the digits from right to left as you pick up the piles and turn them from face up to face down, using the appropriate digit to determine the position of the chosen pile. For example, if the
target position is \(15 = 120_3\), the first time the chosen pile goes on top (none above it), the second time on the bottom (two above it), the third time in the middle (one above it, one below).

Now count out the face down deck, starting from 0, and turn over the spectator’s card at number 15.∗

Because the count starts at 0 the middle card in the deck is number \(13 = 111_3\). Those three 1’s tell you why the chosen pile always goes in the middle in the original version of the trick. In Gergonne’s analysis, repeated by Rouse Ball and others, counting starts at 1. Then the middle card is at number 14, and the discussion of the generalization is cluttered with mysterious 1’s to be added and subtracted.† Counting from 0 makes the connection with base three arithmetic much clearer, and makes a nice piece of patter for the budding magician.

Radix sort
The previous sections described how to do the trick. The question Why does it work? has several answers. What’s new about the one that follows is the connection to radix sort, a well known algorithm for putting things in order, for example, in a computer [7, pp. 170–173].

Suppose you shuffle a deck of 27 cards numbered 0, 1, . . . , 26. (Counting from 0 is standard practice in computer science.) To restore them to numerical order:

- Express each of the card values in base three.
- Deal the cards into three piles labelled 0, 1, and 2, putting each card in the pile that matches its rightmost digit. Pick up the cards with pile 0 on top, then pile 1, then pile 2 on the bottom, preserving the order of the cards in each pile.
- Repeat, this time using the middle digit to place the cards in piles.
- Repeat, using the leftmost digit.

The cards are in order. To see why, note that after the first pass the cards with numbers that end in 0 are above those that end in 1, which are in turn above those that end in 2. As you deal the next pass, they retain that partial order in each pile of nine, so, for example, the cards in pile 0 are the ones with numbers ending in “00”, “01” and “02” in that order (whatever their leftmost digits). The final pass sorts by the leftmost digit.

Think back to Gergonne’s trick. Only the card the spectator chose has a prescribed position in the “sorted” deck. So when you deal out the cards you need not assign them to labelled piles as in radix sort, you just play them as they come. After the deal you label the single pile the spectator identifies with the appropriate digit 0, 1 or 2. Put that one in its proper place as you pick up the piles.

When I teach kids this trick they find the base three expansion of the desired final position of the card by taking out as many nines as they can (0, 1 or 2), then as many threes as possible from the remainder. What’s left is the units digit. The standard computer algorithm cleverly finds the digits in the opposite order, from right to left, but this way is easier for kids to grasp.

Generalizing the trick and the sort
Gergonne knew he could do his trick with a deck of \(n^n\) cards but only \(n = 3\) is practical: \(2^2\) is a trivial 4, and \(4^4 = 256\) is too many cards to handle. Martin Gardner discusses—in principle only—magician Mel Stover’s gargantuan \(10^{10} = \) ten billion card trick, dealing ten times into ten piles of a billion cards each [4, ch. 3], [8, p. 21].

*Before you try teaching this to children, work it for yourself several times. Just reading the description doesn’t educate your hands. In fact I find I that mine forget the manipulations after a while.

†Some authors finesse the problem by counting from the bottom of the deck.
But radix sort and hence Gergonne’s trick will work with an $n^k$ card deck for any $k$. Since the numbers from 0 to $n^k - 1$ have $k$ digit base $n$ expansions you will need $k$ passes to move the selected card to the selected place. In particular, decks of 8, 16 and 32 cards work well to teach binary notation. Expand the target position as a string of three or four or five zeroes and ones. At each pass deal the cards into just two piles and use the digits from right to left to determine which pile to pick up first.

In fact, radix sort works even when the size of the deck isn’t a power of the base. $k$ passes will do the job for $m$ cards when $n^{k-1} < m \leq n^k$. For Gergonne’s trick the piles must be the same height, so $m$ must be a multiple of $n$. A great-nephew of mine showed me a version with $m = 21$ and $n = k = 3$: the piles are seven cards high. Since there are 27 three digit numbers in base three that you can use to specify the position of the designated pile in each pass and only 21 positions, it takes some work to see precisely how the final position of the selected card depends on the order in which you pick up the piles. For some of the three digit strings that position depends on where the card started. For example, you can show that if the selected card is in position 0, 1 or 2 then “001” moves it to position 0 (the top of the deck) but any other selected card ends up in position 1. Of course “000” moves any selected card to the top of the deck. But there is no single three digit string that can move any selected card to position 1.

The following table shows the good target positions, to which you can move every card, and the base three string (to be read from right to left) that tells you how to pick up the piles.

<table>
<thead>
<tr>
<th>position</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>7</th>
<th>10</th>
<th>12</th>
<th>13</th>
<th>17</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>code</td>
<td>000</td>
<td>010</td>
<td>011</td>
<td>022</td>
<td>100</td>
<td>111</td>
<td>122</td>
<td>200</td>
<td>211</td>
<td>212</td>
<td>222</td>
</tr>
</tbody>
</table>

Note that “111” is still the code for the middle (10th) position, even though it represents 13 in base three.

When $m = 12$ every target position is good, and often you have a choice of how to get there, which may make the trick a little less transparent. But there’s no middle to the deck: “111” moves some selected cards to position 5 and others to position 6.

I leave to the reader the verification of these tables and the formulation and proof of any theorems they suggest. Finding them yourself will be more fun than searching for them in the literature.

There’s a generalization of radix sort that points to more Gergonne tricks. Suppose you wish to sort a shuffled full deck of 52 cards so that they end up in the order

$$
\spadesuit A, \spadesuit 2, \ldots, \spadesuit K, \heartsuit A, \heartsuit 2, \ldots, \heartsuit K, \diamondsuit A, \diamondsuit 2, \ldots, \diamondsuit K, \clubsuit A, \clubsuit 2, \ldots, \clubsuit K
$$

Simply deal first into 13 piles of 4 cards, one for each of the values A, 2, \ldots, K. Pick up the piles in order. Then deal into four piles of 13, one for each suit. Finally, assemble the piles in the suit order you wish. This amounts to describing the cards using a mixed radix in which the “units” digit is one of the thirteen possible card values and the “tens” digit is the suit.

Using a mixed radix you can do Gergonne’s trick with a deck of, say, 15 cards, in two passes rather than three by labelling the positions $p$ between 0 and 14 with digit strings “$xy$” where $0 \leq x \leq 4$ and $0 \leq y \leq 2$ so that $p = 3x + y$. First deal the cards into three piles of five, then into five piles of three. Use the digits $y$ and then
x to determine how many piles to put above the selected pile in each pass. Then you
can do the trick the other way, with piles of three followed by piles of five—write
\( p = 5x + y \) with \( 0 \leq x \leq 2 \) and \( 0 \leq y \leq 4 \). Dickson [3] and Onnen[9] wrote about
this generalization.

When you’ve mastered it, move on to a mixed radix with three-digit numbers. But
if you are serious about doing mixed radix Gergonne tricks, it pays to learn the right
to left digit algorithm for expressing numbers in the strange bases you invent.

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Summary  Gergonne’s three pile card trick has been a favorite of mathematicians for nearly two centuries.
This new exposition uses the radix sorting algorithm well known to computer scientists to explain why the trick
works, and to explore generalizations. The presentation suggests strategies for introducing the trick and base three
arithmetic to elementary school students.

A GM-AM Ratio

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Let \( GM(a_1, \ldots, a_n) \) and \( AM(a_1, \ldots, a_n) \) denote the geometric and arithmetic mean
of positive real numbers \( a_1, \ldots, a_n \), respectively. Kubelka [1] proves that for any
\( s > 0 \),

\[
\lim_{n \to \infty} \frac{GM(1^s, \ldots, n^s)}{AM(1^s, \ldots, n^s)} = \frac{s + 1}{e^s}.
\]

He uses the squeeze theorem and a Riemann sum argument. In pursuing a sim-
pler method to show (1), I realized that \( \ln[GM(1^s, \ldots, n^s)/n^s] \) is a Riemann sum