False Position, Double False Position and Cramer's Rule

Eugene C. Boman

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Eugene Boman (ecb5@psu.edu) received his Ph.D. from the University of Connecticut in 1993. While in graduate school he began collecting antique slide rules, a hobby which he continued to pursue after he was hired at Penn State's DuBois campus in 1996 and continues today at Penn State's Harrisburg campus which he joined in 2006. He hopes someday to own enough slide rules to require adding a room to his home in which to store his collection. His wife and two daughters are very patient.

> It is a true saying that "One falsehood leads easily to another."
>
> Cicero (106 BC–43 BC)

> I'm not the greatest; I'm the double greatest. Not only do I knock 'em out, I pick the round.
>
> Muhammad Ali (1942 AD–)

False Position

The Ahmes papyrus, also known as the Rhind papyrus, is essentially a textbook whereby the Egyptian scribe Ahmes taught students the mysteries of the known mathematics of his time, such as it was. Consider the following problem taken from the papyrus [**5**]:

> "Brickmaker, I am in a hurry to erect this house. Today is cloudless, and I do not require many more bricks, for I have all I want but three hundred. Thou alone in one day couldst make as many, but thy son left off working when he had finished two hundred, and thy son-in-law when he had made two hundred and fifty. Working all together in how many days can you make them?"

In the 21st century, once we get past the arcane phrasing, the problem is a triviality. In 1650 BC, not so much.

We would expect our students to solve it thus: Let x be the time required. Then

$$300x + 200x + 250x = 300, \tag{1}$$

so

$$x = \frac{300}{300 + 200 + 250} = 2/5 \text{ of a day.}$$

Lacking access to more than three thousand years of notational development Ahmes and his students would approach the problem in three steps.

1. Guess a solution: Say it takes 2 days to make the 300 bricks we need.
2. Check the guess: In two days the brickmaker makes 600 bricks, his son makes 400 and his son-in-law makes 500 for a total of 1500 bricks. Obviously this guess is way off the mark. No matter. We did not estimate; we guessed.
3. Adjust the guess: Notice that 1500 is exactly five times the desired 300 bricks. Adjusting our guess in the same proportion gives 2/5 days which as we saw is the correct solution.

This is the method of False Position or, when we're showing off our Latin, *Regula Falsa* [**3**], and it is based on simple proportions. In Figure 1 suppose that the length of OC is the desired time; OA is our guess: 2; BA is the number of bricks generated by our guess: 1500; DC is the actual amount desired: 300; and that CD is parallel to AB. Then $OC = \frac{DC}{BA} OA = \frac{300}{1500} 2 = 2/5$.

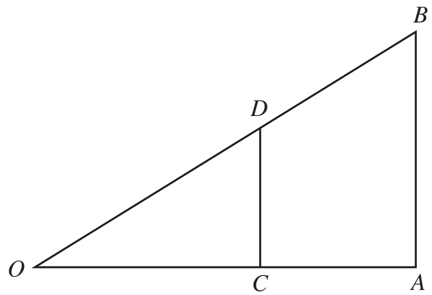

Figure 1. Proportion for the method of False Position.

Used by the Egyptians as far back as 1650 BC, False Position also appears in the "Chiu Chang Suan Shu" (Nine Chapters on the Mathematical Art) of ancient China where it is called *Tshuifen* (Distribution By Progressions) [**10**]. The "Chiu Chang Suan Shu" was transcribed from memory some time after 213 BC when China's emperor Qin Shi Huang burned all books and buried scholars alive [**8, 9**]. The extant version, dates from about 250 AD and is a commentary by Liu Hui [**10**]. Tradition says that the original was completed during the 27th century BC [**8**]. Scholarship only places it in the 3rd century BC [**10**].

False Position also figures prominently in the twelfth chapter of Fibonacci's greatest work, "Liber Abaci" (Book of Calculations) [**6**]. It is possible, though by no means certain, that the method took a more or less direct path from either China or Egypt, to India and on to the Arabic mathematicians from whom Fibonacci learned it. It is also possible that the method was developed independently at all of these times and locations. False Position is so clearly correct intuitively that it is routinely rediscovered by high school and middle school students.

Both the modern algebraic approach and the method of False Position are meant to be entirely mechanical. They can be successfully employed even when not understood, though obviously it is best if we understand what we are doing.

Double False Position

Let's tweak Ahmes's problem a bit. Suppose the customer requires 350 bricks but has obtained 50 already from another source. In modern notation this new problem

$$300x + 200x + 250x + 50 = 350 \quad (2)$$

is easily recognized as equivalent to the original. That is, a modern student would first subtract 50 from both sides, reducing equation 2 to equation 1. In modern notation this problem is barely distinquishable from the original.

However for Ahmes and his students this problem has a new, higher level of complexity. False Position will not work. Try it and see, but be careful that you use False Position exactly as stated. It is very easy to subtract 50 implicitly, thereby fooling yourself into thinking that it does work.

Indeed, it is so very clear to the modern reader that the two problems are equivalent that it is probably difficult to believe that Ahmes and his students did not see this as well. Perhaps some of them did. There are after all, geniuses in every age. Nevertheless, we can safely conclude that Ahmes, his students and other ancient mathematicians would not have done what is so obvious to us precisely because a different solution scheme was developed and commonly used for problems like this one.

Ahmes and his students would proceed as follows:

1. Guess two solutions: Say, $g_1 = 1$ and $g_2 = 2$.
2. Compute the corresponding errors: After 1 day the brickmaker, his son, and son-in-law have made 750 bricks. Combine this with the 50 obtained from another source and we have 800, which is 450 too many. So our first error, e_1, is 450. After 2 days they have 1200 too many so $e_2 = 1200$.
3. Compute the solution thus: $x = \frac{g_1 e_2 - g_2 e_1}{e_2 - e_1}$. Since they didn't have access to modern algebraic symbolism they would presumably compute the solution via some mantra like "First guess times second error minus the second guess times first error all divided by the difference of the errors."

This is the method of Double False Position or *Regula Duorum Falsorum* [**3**]. To see that it works, suppose in Figure 2 that line segment *DK* is parallel to segment *FH*; the length of OA is our solution x; AB is 350; OC is g_1; CD is the number of bricks generated by g_1; OE is g_2; EF is the number of bricks generated by g_2. Then $FH = x - g_2$, $DK = x - g_1$, $BH = e_2$, and $BK = e_1$. Since triangle *DKB* is proportional to triangle *FHB* we see that $\frac{x-g_1}{e_1} = \frac{x-g_2}{e_2}$. Solving for x gives $x = \frac{g_1 e_2 - g_2 e_1}{e_2 - e_1}$ as required. The relationship between False Position and Double False Position should be

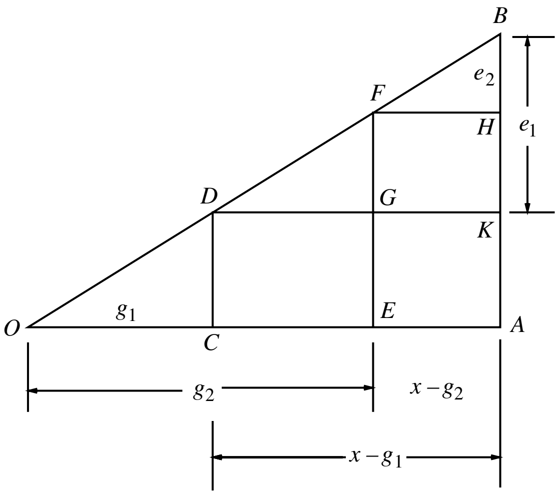

Figure 2. Proportions for the method of Double False Position

clear. False Position requires a single proportion; Double False Position, appropriately, requires two.

This is *not* a reduction to the previously solved problem. Ahmes and his students recognize that they are dealing with a more general problem so they use a more general technique. Notice that Double False Position also works wherever False Position does. Try it on the original problem and see.

Double False Position also appears in "Chiu Chang Suan Shu" as *Ying Buzu Shu* (Method of Surplus and Deficiency) [**8**], where, in addition to solving problems like Ahmes's, it is used iteratively to approximate the roots of polynomial equations. Indeed, as indicated by Maruszewski elsewhere in this issue [**2**] with a little effort it is possible to derive Newton's Method from Double False Position [**8, 7**], once the derivative has been defined.

And Cramer's Rule

Fibonacci describes Double False Position in the 13th chapter of "Liber Abaci" where he calls it by its Arabic name: *elchataym* and says that this is the method "by which the solutions of nearly all problems are found" [**6**], a slight overstatement, perhaps to be attributed to the excitement of discovery. Fibonacci's use of this method is the subject of recent speculation in this *Journal* by Brown and Brunson [**1**] and Maruszewski [**2**]. He also invented a visual mnemonic using "St. Andrew's Cross" ($\times$) to indicate quantities to be multiplied and then subtracted. The following is reminiscent of the one used by Fibonnacci [**4**]. The extremities of the Cross indicate quantities to be multiplied and subtracted in the obvious fashion.

$$x = \frac{\begin{matrix} g_1 & & e_1 \\ & \times & \\ g_2 & & e_2 \end{matrix}}{\begin{matrix} 1 & & e_1 \\ & \times & \\ 1 & & e_2 \end{matrix}}. \tag{3}$$

Upon close inspection Cramer's Rule is recognizably present in this formula but there is no system of linear equations in *obvious* evidence. Whence Cramer's Rule?

We saw in Figure 2 that $\frac{x-g_1}{e_1}$ and $\frac{x-g_2}{e_2}$ are equal. Let their common ratio be y so that

$$\frac{x-g_2}{e_2} = y,$$

and

$$\frac{x-g_1}{e_1} = y.$$

The above system of equations is equivalent to

$$\begin{cases} x - e_2 y = g_2 \\ x - e_1 y = g_1. \end{cases}$$

Cramer's Rule gives $x = \frac{g_1 e_2 - g_2 e_1}{e_2 - e_1}$, as required.

The astute reader will no doubt have noticed that although Cramer's Rule works, in order to get the solution in the same form as that given by Double False Position, we must factor out a negative one from both the numerator and the denominator. Regrettable, but true. However, this would not have troubled Ahmes at all. He did not believe that a larger quantity could be deducted from a smaller one, so he would simply have arranged for all of the subtractions to yield a positive number.

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Perelman's reluctance to make a big deal of his work does not stem from arrogance. As far as Poincare's Conjecture is concerned, those to whom Perelman aimed his papers would know what they imply, and those who would not know should not be reading his papers in the first place. So why make a fuss about the conjecture? [p. 217]

The papers were written for experts in geometric analysis, not for topologists. And even they would have to invest great effort to study them. In fact, it was to take eighteen man-years to figure out whether everything was correct. [p. 219]

Many topologists feel a wistful sadness, however, a sort of postpartum depression. The great adventure that had seen so many ups and downs since its inception in 1904, had kept hundreds of mathematicians busy for a century, and had made and nearly ruined many a career has now, finally, come to an end. [p. 246]

—George G. Szpiro, *Poincare's Prize*,
reviewed on page 310