Subsequence Rational Ergodicity of Rank-One Transformations

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Weak Rational Ergodicity
- We consider standard Borel measure spaces, denoted $(X, B, \mu)$, where $\mu$ is a nonatomic $\sigma$-finite measure. We are interested in the case when $\mu$ is infinite.
- An invertible transformation $T: X \to X$ is said to be measurable if $T(A) \in B$ for all $A \in B$, and is said to be measure-preserving if $\mu(A) = \mu(T(A))$ for all such $A$.
- An invertible transformation $T: X \to X$ is said to be ergodic if $T^n(A) = A$ implies $\mu(A) = 0$ or $\mu(A) = 1$.
- Given an invertible, measure-preserving transformation $T$ and a set $F \subseteq X$ of positive finite measure, define the intrinsic weight sequence of $F$ to be $\nu(F) = \mu(T^n(F))/\mu(F)$. For the sum of these weights up to $n$ we write $\nu_n(F) = \sum_{k=0}^{n-1} \nu_k(F)$.

### A set $F$ is said to sweep out $\mu(X)$ (up to $1$) if
\[ \mu(X) \leq \sum_{k=0}^{n-1} \nu_k(F). \]

Furthermore, we say that $T$ is boundedly rationally ergodic if there exists an $F \subseteq X$ of positive finite measure that sweeps out such that
\[ \limsup_{n \to \infty} \frac{\nu_n(F)}{\mu(X)} \leq 1. \]

If (2) holds only for a subset $(a_n) \subseteq \mathbb{N}$, then we say that $T$ is subsequence boundedly rationally ergodic.

If $F$ is finite, then any invertible ergodic transformation $T$, $X \to X$ is boundedly rationally ergodic. This is not true if $\mu$ is infinite.

#### Theorem (Aaronson ’79)
Bounded rational ergodicity implies weak rational ergodicity. Consequently, subsequence bounded rational ergodicity implies subsequent weak rational ergodicity.

### Rank-One Transformations

#### We take a first column
- We consider standard Borel measure spaces, denoted $(X, B, \mu)$, where $\mu$ is a nonatomic $\sigma$-finite measure. We are interested in the case when $\mu$ is infinite.
- An invertible transformation $T: X \to X$ is said to be measurable if $T(A) \in B$ for all $A \in B$, and is said to be measure-preserving if $\mu(A) = \mu(T(A))$ for all such $A$.
- An invertible transformation $T: X \to X$ is said to be ergodic if $T^n(A) = A$ implies $\mu(A) = 0$ or $\mu(A) = 1$.
- Given an invertible, measure-preserving transformation $T$ and a set $F \subseteq X$ of positive finite measure, define the intrinsic weight sequence of $F$ to be $\nu(F) = \mu(T^n(F))/\mu(F)$. For the sum of these weights up to $n$ we write $\nu_n(F) = \sum_{k=0}^{n-1} \nu_k(F)$.

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### Rank-One Transformations Cont’d

- A rank-one transformation is invertible, ergodic, measure-preserving and every positive finite measure set sweeps out the space

#### Proof of Theorem 1 Cont’d

\[
\sum_{k=0}^{n-1} \frac{\mu(T^n(F))}{\mu(F)} \leq \sum_{k=0}^{n-1} \nu_k(F) \leq 1
\]

Setting $\epsilon = \mu(F)/\mu(X)$, we satisfy the condition in (4).

### Centralizer of a Rank-One Transformation

#### Let $T$ be a conservative ergodic measure-preserving transformation. If $T$ is subsequence weakly rationally ergodic and $S$ is an invertible, nonmeasurable transformation in $C(T)$, then $S$ is measure-preserving.

### Sketch of the Proof

- It is well-known that if $S$ is nonmeasurable and comes with $T$, then there exists a $c$ such that $\mu(S) = c \cdot \mu(A)$ for all measurable $A \subseteq X$. Since $T$ is subsequence weakly ergodic, there exists a set $F \subseteq X$ that sweeps out and a sequence $(a_n) \subseteq \mathbb{N}$ on which for all $A \subseteq F$

#### Theorem 2

Let $T$ be a rank-one transformation and $I$ be the level in $C_{\mu}$. If for all $B \subseteq I$ and for a fixed $n$, we have that
\[ \frac{1}{\nu_n(F)} \sum_{i=0}^{n-1} \nu_i(T^n(F)) \leq c \cdot \mu(B), \]

Then for each $n$, \[ \frac{1}{\nu_n(F)} \sum_{i=0}^{n-1} \nu_i(T^n(F)) \leq c \cdot \mu(B). \]

### Proof of Theorem 2

We fix an $\epsilon$ in our space and show that

\[
\frac{1}{\nu_n(F)} \sum_{i=0}^{n-1} \nu_i(T^n(F)) \leq \epsilon
\]

We notice that it suffices to show (6) holds for $\epsilon = 1$. For a $\mu$-almost every $\epsilon$, we have that

\[
\frac{1}{\nu_n(F)} \sum_{i=0}^{n-1} \nu_i(T^n(F)) \leq c \cdot \mu(B).
\]

### Main Lemma

- Let $T$ be a rank-one transformation and $I$ be the level in $C_{\mu}$. Then the sets $\{T^n(I)\}$ for $0 \leq n \leq \epsilon$ cover almost every point of $I$ between $[I(D, m)]$ and $[I(D, m + 1)]$ times.

### Proof of Theorem 1

By the previous lemma, since the sets $\{T^n(I)\}$ for $0 \leq n \leq \epsilon$ cover almost every point of $I$ at most $2 \cdot [I(D, m)]$ times and at least $\lceil [I(D, m)] \rceil$ times, we get that

#### Further Questions

- We strongly believe that there should exist rank-one transformations with unbounded cuts that are weakly ergodic.
- Also, an example of a rank-one transformation that is not weakly ergodic would be very interesting to find. It should definitely have unbounded cuts by our theorem above.
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