Subsequence Rational Ergodicity of Rank-One Transformations

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Abstract

We show that all rank-one transformations are subsequence boundedly rationally ergodic and that there exist rank-one transformations that are not weakly rationally ergodic.

Weak Rational Ergodicity

- We consider standard Borel measure spaces, denoted $(X, \mathcal{B}, \mu)$, where μ is a nonatomic σ-finite measure. We are interested in the case when μ is infinite.
- An invertible transformation $T : X \to X$ is said to be **measurable** if $T(A) \in \mathcal{B}$ for every $A \in \mathcal{B}$, and is said to be **measure-preserving** if $\mu(A) = \mu(T(A))$ for all such A.
- An invertible transformation $T : X \to X$ is said to be **ergodic** if $T^{-1}(A) = A$ implies $\mu(A) = 0$ or $\mu(A^c) = 0$.
- Given an invertible, measure-preserving transformation T and a set $F \subset X$ of positive finite measure, define the **intrinsic weight sequence** of F to be $u_n(F) = \mu(F \cap T^k F)/\mu(F)^2$. For the sum of these weights up to n write

$$a_n(F) = \sum_{k=0}^{n-1} u_k(F). \tag{1}$$

A set F is said to **sweep out** if $\mu\left(X \setminus \bigcup_{i=0}^{\infty} T^i F\right) = 0$.
- Furthermore, we say that T is **boundedly rationally ergodic** if there exists an $F \subset X$ of positive finite measure that sweeps out such that

$$\sup_{n \geq 1} \left\| \frac{1}{a_n(F)} S_n(1_F) \right\|_{\infty} < \infty. \tag{2}$$

- If (2) holds only for a subset $\{n_i\} \subset \mathbb{N}$, then we say that T is **subsequence boundedly rationally ergodic**
- If μ is finite, then any invertible ergodic transformation $T : X \to X$ is boundedly rationally ergodic. This is not true if μ is infinite

Theorem (Aaronson '79)

Bounded rational ergodicity implies weak rational ergodicity. Consequently, subsequence bounded rational ergodicity implies subsequence weak rational ergodicity.

Rank-One Trasformations

- We take a first column C_0 consisting of a single measurable set of positive finite measure
- In each step, given C_n, we cut the column into r_n subcolumns, where $r_n \geq 2$. That is, we divide B_n, the base of column C_n, into r_n sets of equal measure. If we label these sets as $B_{n,0}, B_{n,1}, \dots, B_{n,r_n-1}$, then our first subcolumn would be $B_{n,0}, T(B_{n,0}), \dots, T^{h_n-1}(B_{n,0})$ and our other $r_n - 1$ subcolumns would be defined similarly
- Above any subcolumn, we may add any number of new levels, called **spacers**, under the condition that these new levels are also pairwise disjoint. Then, C_{n+1} is constructed by stacking each subcolumn with its associated spacers under the next subcolumn. If A is a level below a level B in C_{n+1}, then we must have that $T(A) = B$, $\mu(A) = \mu(B)$, and T is invertible on A. Thus, C_{n+1} will consist of r_n copies of C_n possibly separated by spacers
- Given a column C_n of T, we let h_n be the height of the column and r_n be the number of subcolumns that C_n is cut into. Given a level J in C_n, we denote the height of J in C_n by $h(J)$. If we fix J a level in C_n, and an $m \geq n$, we define the **descendants** of J in C_m to be the set of levels in C_m whose disjoint union is C_n. We then define $D(J, m)$ as the set of the heights of these levels
- To form C_{n+1} from C_n, we first cut C_n into r_n subcolumns, which we denote by $C_n[0], C_n[1], \dots, C_n[r_n - 1]$. Then, before stacking, we add $s_{n,k}$ spacers above each $C_n[k]$, $0 \leq k \leq r_n - 1$, where each $s_{n,k}$ is in $\mathbb{Z}_{m \geq 0}$
- Then define $h_{n,k} = h_n + s_{n,k}$ for each k. If we let

$$H_n = \{0\} \cup \left\{ \sum_{k=0}^{i} h_{n,k} : 0 \leq i < r_n - 1 \right\},$$

we have that, for J in C_j and $N \geq j$,

$$D(J, N) = h(J) + H_j \oplus H_{j+1} \oplus \cdots \oplus H_{N-1}.$$

- With this notation we can easily find the number of elements in $D(J, N)$ to be $|D(J, N)| = |H_j| \cdot |H_{j+1}| \cdots |H_{N-1}| = r_j \cdot r_{j+1} \cdots r_{N-1}$. For a level J in C_n and $m \geq n$, we define the maximum height of its descendants in C_m to be $M_m = \max\{D(J, m)\}$

Rank-One Transformations Cont'd

- **A rank-one trasnformation is invertible, ergodic, measure-preserving, and every positive finite measure set sweeps out the space**

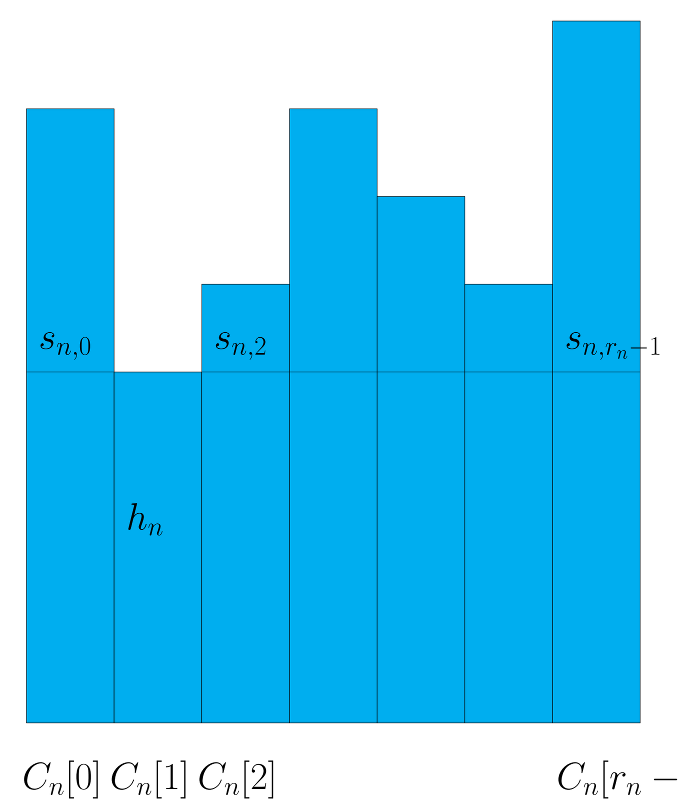

Figure 1: Rank-One Transformation: column C_n after placing the spacers

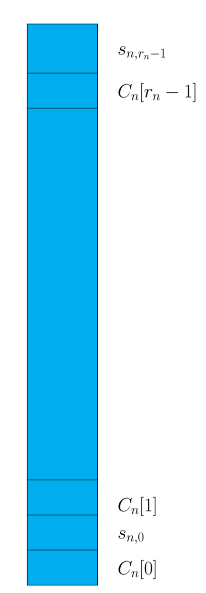

Figure 2: Column C_{n+1} after stacking the subcolumns of C_n

Main Lemma

Let T be a rank-one transformation and I the level in C_0. Then, the sets $\{T^k I\}$ for $|k| \leq M_m$ cover almost every point of I between $|D(I, m)|$ and $2 \cdot |D(I, m)|$ times.

Theorem 1

Let T be a rank-one transformation and I be the level of C_0. Then there exists a constant $c > 0$ such that

$$\sum_{k=0}^{n_m-1} \mu(I \cap T^k B) \leq c\mu(B) \sum_{k=0}^{n_m-1} \mu(I \cap T^k I) \tag{3}$$

for all $B \subset I$, where $n_m = M_m + 1$.

Proof of Theorem 1

By the previous lemma, since the sets $\{T^k I\}$ for $-M_m \leq k \leq M_m$ cover almost every point of I at most $2 \cdot |D(I, m)|$ times and at least $|D(I, m)|$ times, we get that

Proof of Theorem 1 Cont'd

$$\sum_{k=-M_m}^{M_m} \mu(I \cap T^k B_1) \leq 2 \left(\frac{|D(I, m)|}{|D(I, m)|} \right) \left(\frac{\mu(B_1)}{\mu(I)} \right) \left(\sum_{k=-M_m}^{M_m} \mu(I \cap T^k I) \right).$$

So

$$\sum_{k=0}^{M_m} \mu(I \cap T^k B_1) \leq 2 \left(\frac{|D(I, m)|}{|D(I, m)|} \right) \left(\frac{\mu(B_1)}{\mu(I)} \right) \left(2 \left(\sum_{k=0}^{M_m} \mu(I \cap T^k I) \right) - 1 \right)$$

$$\leq 4 \frac{\mu(B_1)}{\mu(I)} \sum_{k=0}^{M_m} \mu(I \cap T^k I).$$

Setting $c = 4/\mu(I)$, we satisfy the condition in (4).

Theorem 2

Let T be a rank-one transformation and I be the level in C_0. If for all $B \subset I$ and for a fixed $n \in \mathbb{N}$, we have that

$$\frac{1}{a_n(I)} \sum_{k=0}^{n-1} \mu(I \cap T^k B) \leq c\mu(B), \tag{4}$$

then for that n,

$$\left\| \frac{1}{a_n(I)} S_n(1_I) \right\|_{\infty} \leq c \tag{5}$$

Proof of Theorem 2

We fix an x in our space and show that

$$\frac{1}{a_n(I)} \sum_{k=0}^{n-1} 1_I \circ T^k(x) \leq c. \tag{6}$$

We notice that it suffices to show (6) holds for $x \in I$. For $x \notin I$, we either have that $T^j(x) = y$ for some $y \in I$, $0 \leq j \leq n-1$ or there are no such y and j. In the former case, we take the least such j and have that $\frac{1}{a_n(I)} \sum_{k=0}^{n-1} 1_I \circ T^k(x) \leq \frac{1}{a_n(I)} \sum_{k=0}^{n-1} 1_I \circ T^k(y) \leq c$, and in the latter we have that $\frac{1}{a_n(I)} \sum_{k=0}^{n-1} 1_I \circ T^k(x) = 0$.

Now, for every $x \in I$, and a fixed ..., we have that there exists a column C_m for which if $J \ni x$ is a level in C_m, there are at least n levels above J. Then, we see that

$$\frac{1}{a_n(I)} \sum_{k=0}^{n-1} 1_I \circ T^k(x) = \frac{1}{a_n(I)} \sum_{k=0}^{n-1} \frac{\mu(I \cap T^k J)}{\mu(J)} \leq c.$$

Now, since (6) holds for all $x \in I$ and therefore all $x \in X$, we have that (6) holds.

As a corollary of Theorem 1 and Theorem 2 we get the following result:

Main Theorem

All rank-one transformations are subsequence boundedly rationally ergodic. Consequently, by Aaronson's theorem we get that all rank-one transformations are subsequence weakly rationally ergodic.

Further Results

If we consider T to be a rank-one transformation with bounded number of cuts, i.e. $sup_n r_n < \infty$, then we can strenghten the main result. Namely, all rank-one transformations with bounded number of cuts are boundedly rationally ergodic.

We also prove that rank-one transformations are generic in the space of invertible mesure-preserving transformations. A result of Aaronson which says that weakly rationally ergodic transformations are meager in the above mentioned space. **Therefore there are rank-one transformations that are not weakly rationally ergodic**. We do not know any examples of this.

Centralizer of a Rank-One Transformation

- $S : X \to X$ is an **invertible nonsingular** transformation if it is invertible, measurable and $\mu(A) = 0$ if and only if $\mu(S(A)) = 0$.
- The (nonsingular) **centralizer** $C(T)$ of a transformation T consists of all invertible nonsingular S such that $ST = TS$.

Theorem (after a theorem of Aaronson '77)

Let T be a conservative ergodic measure-preserving transformation. If T is subsequence weakly rationally ergodic and S is an invertible, nonsingular transformation in $C(T)$, then S is measure-preserving.

Sketch of the Proof

It is well-known that if S is nonsingular and commutes with T, then there exists a c such that $\mu(S(A)) = c \cdot \mu(A)$ for all measurable $A \subset X$. Since T is subsequence weakly rationally ergodic, there exists a set $F \subset X$ that sweeps out and a sequence $\{n_i\} \subset \mathbb{N}$ on which for all $A, B \subset F$,

$$\lim_{i \to \infty} \frac{1}{a_{n_i}(F)} \sum_{k=0}^{n_i} \mu(A \cap T^k B) = \mu(A)\mu(B).$$

By the invertibility of S, for any $C, D \subset SF$, there exist $A, B \subset F$ such that $C = SA, D = SB$. Then, as $\mu \circ S^{-1} = c^{-1}\mu$, and S and T commute,

$$\lim_{i \to \infty} \frac{1}{a_{n_i}(SF)} \sum_{k=0}^{n_i} \mu(C \cap D) = \mu(C)\mu(D)$$

for all $C, D \subset SF$. As the subsequence weak rational ergodicity property holds on F and SF, it also holds on $F \cup SF$ (Aaronson '77), so for $A \subset F$,

$$\mu(SA)\mu(SA) = \lim_{i \to \infty} \frac{1}{a_{n_i}(F \cup SF)} \sum_{k=0}^{n_i} \mu(SA \cap T^k SA)$$
$$= c \cdot \lim_{i \to \infty} \frac{1}{a_{n_i}(F \cup SF)} \sum_{k=0}^{n_i} \mu(A \cap T^k A)$$
$$= c\mu(A)\mu(A).$$

But we also have that $\mu(SA)^2 = c^2\mu(A)^2$, so $c = 1$.

Corollary

T is called **squashable** if it commutes with a non-measure-preserving S. By the Main Theorem and the above theorem we get the following corollary:

Centralizer of Rank-One Transformations

It T is a rank-one transformation and S is in the centralizer of T, then S is measure-preserving. Thus T is not squashable.

Further Questions

We strongly believe that there should exist rank-one trasformations with unbounded cuts that are weakly rationally ergodic.

Also, an example of a rank-one trasnformations that is not weakly rationally ergodic would be very interesting to find. It should definitely have unbounded cuts by our theorem above.

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