[9] and Wells [10] is closely related to the 3-dimensional case of the main theorem. It explains what happens when the six vertices of the octahedron are allowed to lie in the same plane. This result can be generalized to polygons with an even number of sides.

Figures in this note and additional figures can be found at the MAGAZINE website, as well as *Geometer's Sketchpad* or *Cabri 3d* files that allow experimentation.

REFERENCES

1. N. Altshiller-Court, *Modern Pure Solid Geometry*, 2nd ed., Chelsea, Bronx, New York, 1964.
2. H. S. M. Coxeter, *Regular Polytopes*, 2nd ed., Macmillan, New York, 1963. Reprinted by Dover Publications, New York, 1973.
3. H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, Washington, DC, 1967.
4. M. Khosravi and M. D. Taylor, The Wedge Product and Analytic Geometry, *Amer. Math. Monthly* **115** (2008) 623–644.
5. M. F. Mammana, B. Micale, and M. Pennisi, On the centroids of polygons and polyhedra, *Forum Geometricorum* **8** (2008) 121-130.
6. A. Nash, A generalized parallelogram law, *Amer. Math. Monthly* **110** (2003) 52–57.
7. P. N. Oliver, Pierre Varignon and the Parallelogram Theorem, *The Mathematics Teacher* **94** (2001) 316–319.
8. D. M. Y. Sommerville, *An Introduction to the Geometry of N Dimensions*, Dutton, New York, 1929. Reprinted by Dover Publications, New York, 1958.
9. E. W. Weisstein. Centroid Hexagon, *MathWorld–A Wolfram Web Resource*, `http://mathworld.wolfram.com/CentroidHexagon.html`.
10. D. Wells, *The Penguin Dictionary of Curious and Interesting Geometry*, Penguin, London, 1991.

A Curious Way to Test for Primes Explained

DAVID M. BRADLEY
University of Maine
Orono, ME 04469-5752
bradley@math.umaine.edu

In the October 2007 issue of this MAGAZINE [2], Walsh presents a curious primality test, attributed to a mysterious taxi-cab driver. Sensing there must be more to the story, I decided to track down Walsh's cab driver. As it turned out, the cabbie was bemused to learn that her off-hand remark became the subject of a journal article, so I showed it to her.

"That's interesting," she said, "but I had a simpler result in mind, and also a simpler proof." She then proceeded to explain. "Walsh's test is based on the Maclaurin series expansion

$$e^{(x^k/k)} = \sum_{j=0}^{\infty} \frac{(x^k/k)^j}{j!} = \sum_{j=0}^{\infty} \frac{x^{kj}}{k^j j!}. \tag{1}$$

Using this, he defined

$$g_n(x) = \sum_{k=1}^{n-1} e^{(x^k/k)} = \sum_{k=1}^{n-1} \sum_{j=0}^{\infty} \frac{x^{kj}}{k^j j!},$$

for real x and integer $n > 1$, and then computed the nth derivative of g_n at 0 as

$$g_n^{(n)}(0) = \sum_{\substack{k=1 \\ k|n}}^{n-1} \frac{n!}{k^{n/k}(n/k)!}, \tag{2}$$

employing the standard abbreviation $k|n$ for the condition that $n/k \in \mathbb{Z}$.

Walsh's test amounts to the observation that an integer $n > 1$ is prime if and only if $g_n^{(n)}(0) = 1$. Now it's not hard to see that the k^j factor in the denominator of the rightmost sum in (1) plays no role other than to reduce the size of $g_n^{(n)}(0)$ when n is composite. What I actually had in mind is the following:

THEOREM. *For each integer $n > 1$, define the function $f_n : \mathbb{R} \to \mathbb{R}$ by*

$$f_n(x) = \sum_{k=1}^{n-1} e^{(x^k)}.$$

An integer $n > 1$ is prime if and only if the nth derivative of f_n satisfies $f_n^{(n)}(0) = 1$.

Proof. In light of the fact that the Maclaurin series expansion

$$e^{(x^k)} = \sum_{j=0}^{\infty} \frac{x^{kj}}{j!}$$

is valid for all real x and all positive integers k, it follows that if $x \in \mathbb{R}$, then

$$f_n(x) = \sum_{k=1}^{n-1} \sum_{j=0}^{\infty} \frac{x^{kj}}{j!}. \tag{3}$$

Now we could calculate $f_n^{(n)}$ following Walsh [**2**], by repeatedly differentiating term by term, but it seems easier to note that by Taylor's theorem, $f_n^{(n)}(0)$ is equal to $n!$ times the coefficient of x^n in $f_n(x)$. Observe that we get a contribution to the coefficient of x^n in (3) if and only if $n = kj$. We conclude that

$$f_n^{(n)}(0) = \sum_{\substack{k=1 \\ k|n}}^{n-1} \frac{n!}{(n/k)!} = 1 + \sum_{\substack{k=2 \\ k|n}}^{n-1} \frac{n!}{(n/k)!}. \tag{4}$$

If n is prime, then the sum on the right is empty; otherwise it is strictly positive." ■

I then pointed out that the same idea would work if we eliminated not just the k^j in (1), but the $j!$ too. For, if $|x| < 1$ and k is any positive integer, then the formula for the sum of geometric series with ratio x^k gives

$$\frac{1}{1 - x^k} = \sum_{j=0}^{\infty} x^{kj}. \tag{5}$$

If we now define

$$h_n(x) = \frac{1}{n!} \sum_{k=1}^{n-1} \frac{1}{1 - x^k} = \frac{1}{n!} \sum_{k=1}^{n-1} \sum_{j=0}^{\infty} x^{kj} \tag{6}$$

for integer $n > 1$ and real x such that $-1 < x < 1$, then the same reasoning shows that

$$h_n^{(n)}(0) = \sum_{\substack{k=1 \\ k|n}}^{n-1} 1 = \tau(n) - 1, \tag{7}$$

where $\tau(n)$ is the number of positive integer divisors of n. It follows that $h_n^{(n)}(0) = 1$ if and only if n is prime.

The cabbie nodded. "Of course, it would be nice if you could use a single function to test all positive integers n. It's tempting to try something like

$$\sum_{k=1}^{\infty} \frac{1}{1 - x^k},$$

but that diverges if $-1 < x < 1$. But if you look at (5) and (6), you'll see that the $j = 0$ term plays no essential role in the subsequent argument. Dropping it leads us to consider

$$L(x) := \sum_{k=1}^{\infty} \frac{x^k}{1 - x^k} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} x^{kj},$$

which is valid for all real x such that $-1 < x < 1$. Furthermore, the same reasoning as before shows that

$$\frac{L^{(n)}(0)}{n!} = [\text{coefficient of } x^n \text{ in } L(x)] = \sum_{\substack{k=1 \\ k|n}}^{n} 1 = \tau(n), \tag{8}$$

so a positive integer n is prime if and only if $L^{(n)}(0)/n! = 2$."

"But wait a minute," I said. "What you've actually shown is that if $-1 < x < 1$, then

$$\sum_{k=1}^{\infty} \frac{x^k}{1 - x^k} = \sum_{n=1}^{\infty} \tau(n) x^n.$$

This is nothing other than Lambert's generating series for the divisor function [**1**, p. 280]."

The driver then observed that just as (7) and (8) have obvious combinatorial interpretations, so does (4): It counts the number of ways to partition a set of n distinct objects into ordered tuples of equal length less than n. Obviously, this is equal to 1 if and only if n is prime. The question then arose as to whether Walsh's approach also has a combinatorial interpretation. As Walsh himself confirmed [**3**], his $g_n^{(n)}(0)$ (see (2) above) counts the number of permutations of n distinct objects that can be written as a product of pairwise disjoint cycles of equal length less than n. To see this, note that the number of ways to partition kr distinct objects into r sets of size k is

$$\frac{(kr)!}{r!\,(k!)^r}.$$

For each set, the number of ways to form a cycle of size k is $(k-1)!$. Hence, the number of permutations of kr objects that can be written as a product of pairwise disjoint cycles of length k is equal to

$$\frac{(kr)!}{r!} \left(\frac{(k-1)!}{k!} \right)^r = \frac{(kr)!}{r!\,k^r}.$$

Letting $n = kr$ and summing over $1 \le k \le n-1$ such that $k|n$, we get (2).

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REFERENCES

1. L. E. Dickson, *History of the Theory of Numbers*, vol. 1, Washington DC, 1919.
2. Dennis P. Walsh, A curious way to test for primes, this MAGAZINE **80**(4) (October 2007) 302–303.
3. Dennis P. Walsh, Counting homocyclic permutations, preprint.

More on the Lost Cousin of the Fundamental Theorem of Algebra

ROMAN SZNAJDER
Bowie State University
Bowie, MD 20715-9465
rsznajder@bowiestate.edu

In his recent note [**2**], Timo Tossavainen proves what he calls "The Lost Cousin of the Fundamental Theorem of Algebra," which we state as:

EXPONENTIAL THEOREM. *For any integer $n \geq 1$, let $0 < \kappa_0 < \kappa_1 < \cdots < \kappa_n$ and a_j (for $j = 0, \ldots, n$) be real numbers with $a_n \neq 0$. Then the function $f : \mathbb{R} \to \mathbb{R}$,*

$$f(t) = \sum_{j=0}^{n} a_j \kappa_j^t$$

has at most n zeros.

Years ago, I was presented by a friend with a copy of a concise monograph [**1**] (112 pages long) on selected topics in polynomial approximation. In this book, apparently unknown to western readers, the following fact and its proof appear:

GENERALIZED POLYNOMIAL THEOREM. *A function g given by the formula*

$$g(x) = a_0 x^{\alpha_0} + a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n},$$

where $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ are arbitrary real numbers and $a_n \neq 0$, has no more than n roots.

Proof. We proceed by induction on n, noting that for $n = 1$ the statement is obvious. Assume that for some n the claim is true, but for $n + 1$, it is not. Hence, for some real numbers $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1}$ and $a_{n+1} \neq 0$, there is a function

$$g(x) = a_0 x^{\alpha_0} + a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n} + a_{n+1} x^{\alpha_{n+1}},$$

whose number of positive roots is larger than $n + 1$. These roots are identical with the roots of the new function

$$g(x)/x^{\alpha_0} = a_0 + a_1 x^{\alpha_1 - \alpha_0} + \cdots + a_n x^{\alpha_n - \alpha_0} + a_{n+1} x^{\alpha_{n+1} - \alpha_0}.$$

By Rolle's theorem, the derivative of the above function, which has the form

$$b_0 x^{\beta_0} + b_1 x^{\beta_1} + \cdots + b_n x^{\beta_n},$$