Torified Rational Links

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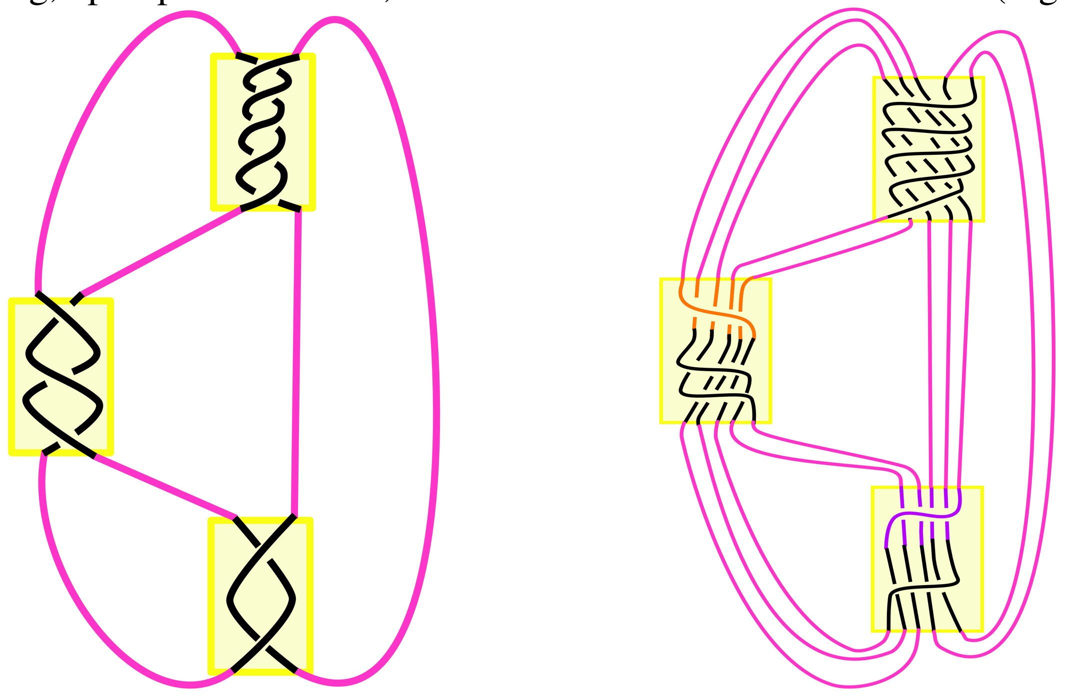
Abstract

We will examine a new class of links, called Torified Rational Links, with a focus on bounding the stick number. These bounds are found by supercoiling the tangles and attaching the coil to an outer skeleton, similar to that of a rational link. The bounds obtained using this model for finding an upper bound are compared to the known upper bound for the stick number of any link, $s(L) \le \frac{3}{2}(c(L)+1)$, using relationships between crossing number and the maximal and minimal degrees of the variables in the HOMFLY polynomial.

Constructing Torified Rational Links

Rational links are a well-studied class of links. They are constructed by stringing integral tangles together in a particular way (Figure 1 a). We are going to construct a new class of links, Torified Rational Links, by swapping out the integral tangles of rational links for a different kind of tangle, while preserving, up to parallel cables, the skeletal structure of rational links (Figure 1 b).



(a) Rational Link (b) Torified Rational Link

Figure 1: How the skeleton (pink) and twist-boxes (yellow) of Rational Links and Torified Rational Links are attached. We define $\sigma(n)$, as moving the rightmost input strand across all other strands. Each twist-box contains a tangle $\sigma(n)^x$ ($\sigma(5)$ shown in purple, $\sigma(5)^{-1}$ shown in orange).

Background

Many invariants can help to distinguish and define links. These include link and knot polynomials, covers, colorings, etc. One such measure is called the stick number.

Definition 1. *The **stick number**, $s(L)$, of a link L is the minimum number of line segments (sticks) needed to represent L. The sticks must be non-intersecting, and can only join at a common vertex.*

Huh and Oh have shown that, for any link L, $s(L) \le \frac{3}{2}(c(L)+1)$, where $c(L)$ is the minimal crossing number [3]. Insko and Trapp improved this bound for sufficiently complex 2-bridge links using the shape and structure of DNA supercoils as a model for a non-minimal crossing polygonal projection (Figure 2) [4].

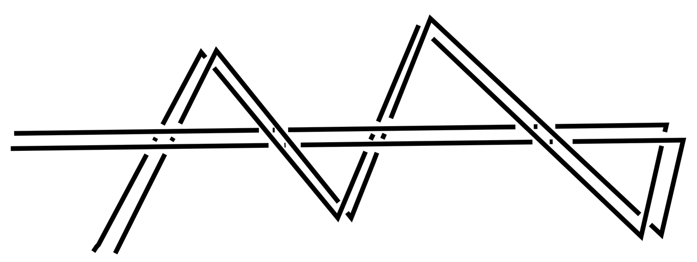
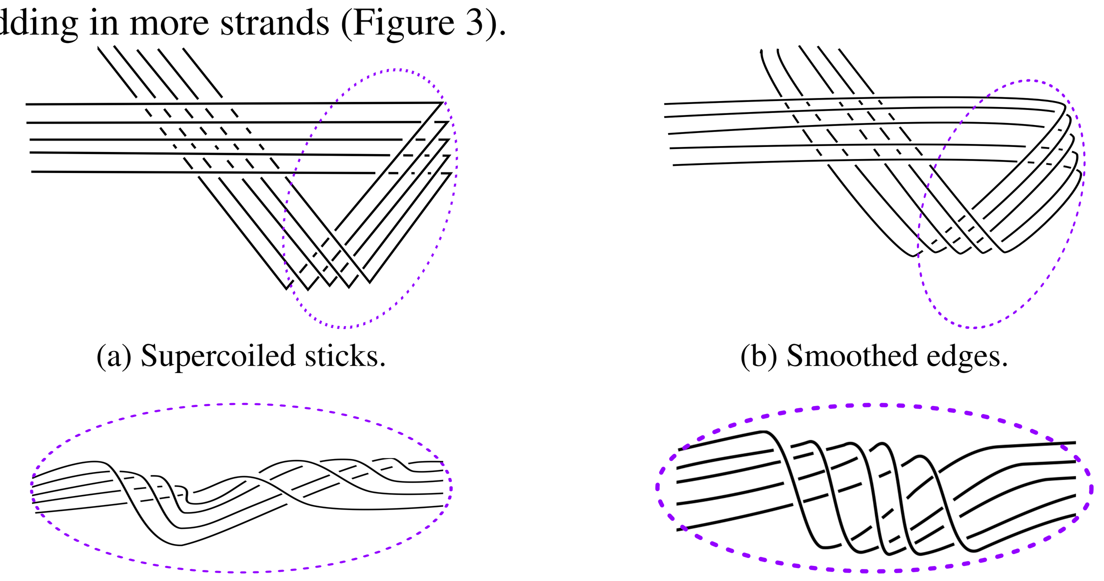


Figure 2: Supercoil of a 2-bridge link's tangle.

We can use a similar model to construct a polygonal representation of Torified Rational tangles by adding in more strands (Figure 3).



(a) Supercoiled sticks. (b) Smoothed edges.

(c) Corner pulled straight. (d) Resulting tangle from corner.

Figure 3: Simple supercoil using five strands.

Results: Links with three twist-boxes

Theorem 1. *Let L be a Torified Rational Link with three twist-boxes T_1, T_2, T_3 containing tangles $\sigma(n_1)^{x_1}, \sigma(n_2)^{x_2}, \sigma(n_3)^{x_3}$, where n_1, n_2, n_3 are the respective box stand numbers. The corresponding k_i are the largest integers such that $x_i = (2+3k_i)n_i + y_i$, and $y_i \ge 0$. Define $s_i(L)$ by:*

$$s_i(L) = \begin{cases} (3+2k_i)n_i & \text{if } y_i = 0 \\ (3+2k_i)n_i + 2(y_i - 1) & \text{if } y_i \le 3n_i(k_i+1) \\ (5+2k_i)n_i + 2(x_i - y_i - 1) & \text{if } y_i > 3n_i(k_i+1). \end{cases} \quad (1)$$

Then the number of sticks needed to build L is bounded by:

$$s(L) \le \min\{4w_1 + 6w_2 - w_3, 5w_1 + 6w_2 - 2w_3\} + \sum_{i=1}^{3} s_i(L). \quad (2)$$

Torified Rational Links that have three twist-boxes are constructed as in Figure 4, where the w_i represent the number of strands along that edge of the link's skeleton.

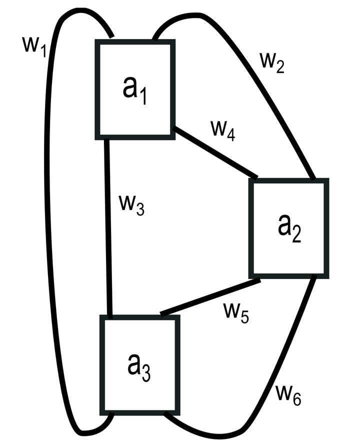


Figure 4

Claim 1. *Weights, $w_1, \ldots, w_6$, are completely determined by w_1, w_2, and w_3.*

We find that $w_5 = w_4 = w_1 + w_2 - w_3$, and $w_6 = w_2$.

Lemma 1. *Then, the number of sticks needed to build the skeleton of a Torified Rational Link, L, with three twist-boxes, $s_s(L)$, is bounded by:*

$$s_s(L) \le 4w_1 + 6w_2 - w_3. \quad (3)$$

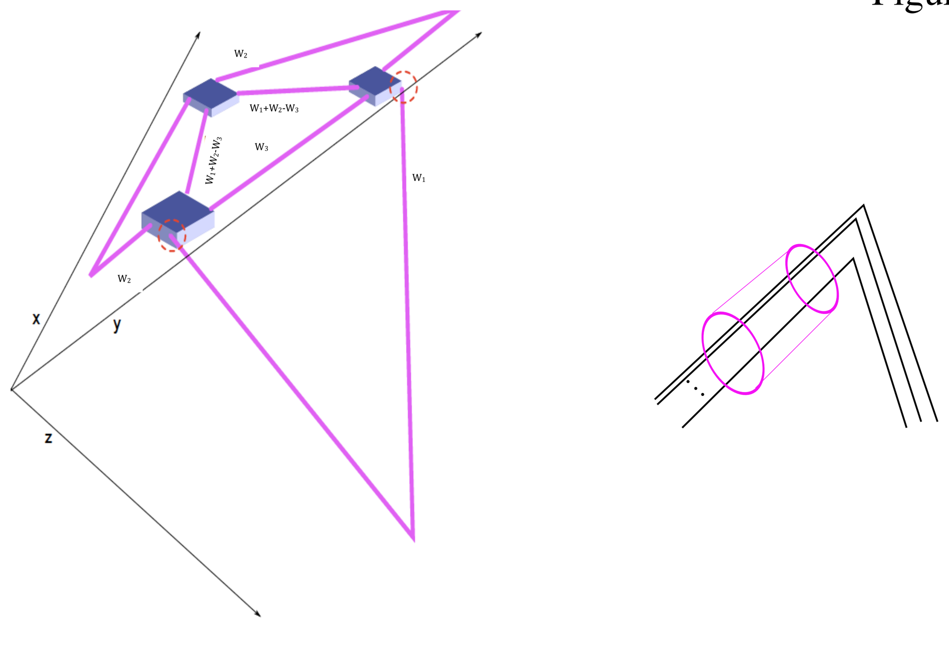


Figure 5: Structure of the skeleton of a Torified Rational Link with three twist-boxes.

Claim 2. *In $\mathbb{R}^3$, we can represent any twist box of n strands containing the tangle $\sigma(n)^x$, where $|x| \ge 1$, with $2(|x|-1)+n$ sticks or fewer.*

Claim 3. *We can represent any Torified Rational tangle, T, with $\sigma(n)^x$ tangles, where $x = (2+3k)n$, using $(3+2k)n$ sticks or fewer, where n is the number of strands and $k \in \mathbb{Z}$.*

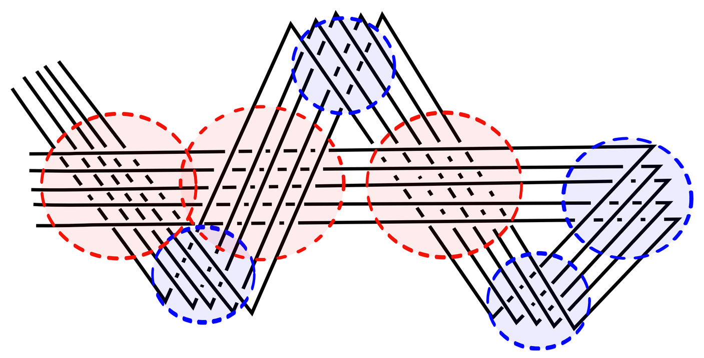


Figure 6: Crossing contribution of a supercoiled Torified Rational tangle.

Lemma 2. *Let T be a tangle of some Torified Rational Link, L. Let T contain the tangle $\sigma(n)^x$. Choose k to be the largest integer such that $x = (2+3k)n + y$, and $y \ge 0$. Then,*

$$s_T(L) \le \begin{cases} (3+2k)n & \text{if } y = 0 \\ (3+2k)n + 2(y-1) & \text{if } 0 < y \le 3n(k+1) \\ (5+2k)n + 2(x-y-1) & \text{if } y > 3n(k+1). \end{cases} \quad (4)$$

Claim 4. *We can construct any Torified Rational tangle such that it is contained entirely in a twist-box, where each strand will only intersect the box at its exit and entry points, which are on opposite faces of the box.*

Claim 5. *Let L be a three-box Torified Rational Link. If all of the twist-boxes of L have supercoils, then we can always attach the skeleton to the twist-boxes by adding only $2w_3 - 2w_2$ new sticks.*

Results: Efficiency of construction

To compare our bound on stick number to that of Huh and Oh, we use Gruber's bound for $c(L)$, which can be derived from the HOMFLY polynomial. That is, for every link L,

$$c(L) \ge M + \frac{1}{2}(E - e), \quad (5)$$

where M (resp. E) is the maximal non-zero exponent on z (resp. v), and e is the minimal non-zero exponent on v [1].

The HOMFLY polynomial of the link in Figure 7 yields $M = 18$, $E = -2$, and $e = -24$. This means that $c(L) \ge 29$, and therefore Huh and Oh's bound on stick number is, at best, $s(L) \le 45$. However, using our bound, we get $s(L) \le 35$, which is significantly better.

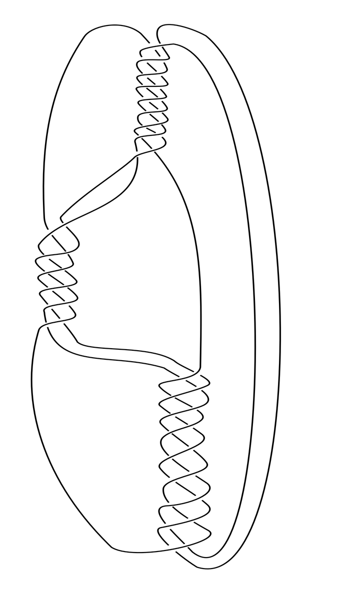


Figure 7

Results: Links with n twist-boxes

Definition 2. *The **box strand number**, n_b, is the number of strands going into the twist-box.*

Claim 6. *Let L be a Torified Rational Link with m twist-boxes. Let the w_i be the weights of the strands as in Figure 8. Then $w_1 = w_{2m}$.*

Claim 7. *For any link with m twist-boxes, $n_1 = n_m$.*

Claim 8. *Let a_T be any Torified Rational tangle of a Torified Rational Link, L, whose first (top) twist-box has box strand number n_1. Then $n_T \le 2n_1 - 2$.*

Definition 3. *Define the **pair strands**, w_{l_i}, to be those strands which are intersected more than once by red lines, excluding w_1, w_2, and w_{2m}.*

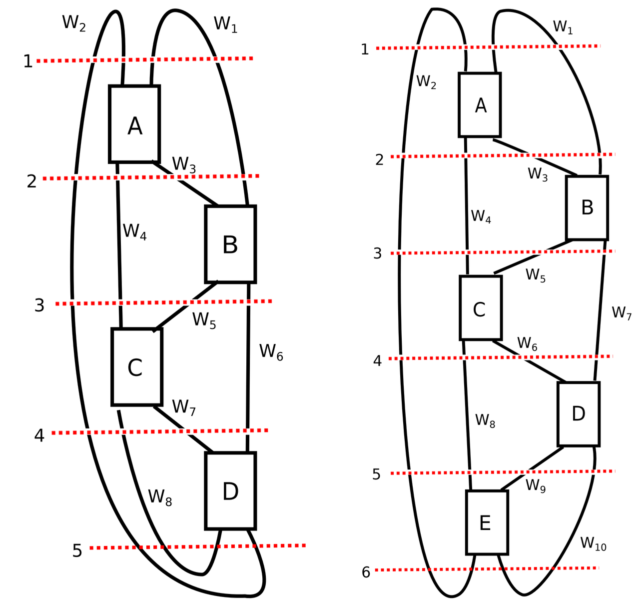


Figure 8

Lemma 3. *Let L be a Torified Rational Link with m twist-boxes, and let $\lambda = \sum_{i=1}^{m-2} w_{l_i}$. Then the number of sticks needed to construct the skeleton of L is bounded by:*

$$s_s(L) \le m(2n_1 - w_2) - \lambda + w_2. \quad (6)$$

Theorem 2. *Let L be a Torified Rational Link with m twist-boxes containing tangles $T_1, T_2, \ldots, T_m$. Suppose each T_i contains the tangle $\sigma(n_i)^{x_i}$, where n_i is the box strand number. Choose k_i to be the largest integer such that $x_i = (2+3k_i)n_i + y_i$, and $y \ge 0$. Let $\lambda = \sum_{i=1}^{m-2} w_{l_i}$. Define $s_i(L)$ as in Equation 1. Then the stick number of L is bounded by:*

$$s(L) \le m(2n_1 - w_2) - \lambda + w_2 + \sum_{i=1}^{m} s_i(L). \quad (7)$$

Open Questions

1. How could we generally prove that our bound on stick number is better than $s(L) \le \frac{3}{2}(c(L)+1)$ using the HOMFLY polynomial?
2. Other than rational and torified rational tangles, are there other types of tangles that can be polygonally represented using supercoils?

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