Torified Rational Links

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Abstract

We will examine a new class of links, called Torified Rational Links, with a focus on bounding the stick number. These bounds are found by supertwisting the tangles and attaching the coil to an outer skeleton, similar to that of a rational link. The bounds obtained using this model for finding an upper bound are compared to the known upper bound for the stick number of any link, \( s(L) \leq 2\sigma(L) + 1 \), using relationships between crossing number and the maximal and minimal degrees of the variables in the HOMFLY polynomial.

Constructing Torified Rational Links

Rational links are a well-studied class of links. They are constructed by stringing integral tangles together in a particular way (Figure 1 a). We are going to construct a new class of links, Torified Rational Links, by swapping out the integral tangles of rational links for a different kind of tangle, while preserving, up to parallel cables, the skeletal structure of rational links (Figure 1 b).

Many invariants can help to distinguish and define links. These include link and knot polynomials, degree of the variables in the HOMFLY polynomial. Finding an upper bound are compared to the known upper bound for the stick number of any link, \( s(L) \leq 2\sigma(L) + 1 \), using relationships between crossing number and the maximal and minimal degrees of the variables in the HOMFLY polynomial.

Results: Links with three twist-boxes

**Theorem 1.** Let \( L \) be a Torified Rational Link with three twist-boxes \( T_1, T_2, T_3 \) containing tangles \( \sigma(a_1)^{b_1}, \sigma(a_2)^{b_2}, \sigma(a_3)^{b_3} \), where \( a_1, \sigma(a_2) \), and \( \sigma(a_3) \) are the respective box stand numbers. The corresponding \( k_i \) are the largest integers such that \( x_i = (2 + 3k_i)n_i + y_i \) and \( y_i \geq 0 \). Define \( s_i(L) \) by:

\[
\begin{align*}
\sigma(a_1)^{b_1} & \leq (2k_1 + 2m_1) + 2(2y_1 - 1) \\
\sigma(a_2)^{b_2} & \leq (2k_2 + 2m_2) + 2(2y_2 - 1) \\
\sigma(a_3)^{b_3} & \leq (2k_3 + 2m_3) + 2(2y_3 - 1)
\end{align*}
\]

Then the number of sticks needed to build \( L \) is bounded by:

\[
s(L) \leq s_1(L) + s_2(L) + s_3(L) - 1
\]

Figure 1: How the skeleton (pink) and twist-boxes (yellow) of Rational Links and Torified Rational Links are attached. We define \( \sigma(n) \), by moving the rightmost input strand across all other strands. Each twist-box contains a tangle \( \sigma(n) \) shown in purple, \( \sigma(n)^{-1} \) shown in orange.

Results: Efficiency of construction

To compare our bound on stick number to that of Hub and Oh, we use Graber’s bound for \( s(L) \), which can be derived from the HOMFLY polynomial. That is, for every link \( L \),

\[
s(L) \leq M + \frac{E}{E - c - 3},
\]

where \( M \) (resp. \( E \)) is the maximal non-zero exponent on \( z \) (resp. \( x \)), and \( c \) is the minimal non-zero exponent on \( x \). The HOMFLY polynomial of the link in Figure 7 yields \( M = 18 \), \( E = -2 \), and \( c = -24 \). This means that \( s(L) \geq 29 \), and therefore Hub and Oh’s bound on stick number is, at best, \( s(L) \leq 25 \). However, our bound, we get \( s(L) \leq 35 \), which is significantly better.

Results: Links with \( n \) twist-boxes

**Definition 2.** The box strand number, \( w_n \), is the number of boxes going into the twist-box.

**Claim 6.** Let \( L \) be a Torified Rational Link with \( n \) twist-boxes. Let the \( w_n \) be the weights of the strands as in Figure 8. Then \( w_1 = w_2 \)

**Claim 7.** For any link with \( n \) twist-boxes, \( w_1 = w_2 \)

**Claim 8.** Let \( \lambda \) be any Torified Rational tangle of a Torified Rational Link, \( L \), whose first (top) twist-box has box strand number \( s \). Then \( s \leq 2w_2 \)

Figure 5: Simple supercoil using five strands.

**Figure 6:** Crossing contribution of a supercoiled Torified Rational Tangle.

**Lemma 2.** Let \( T \) be a tangle of some Torified Rational Link. Let \( T \) contain the tangle \( \sigma(a)^{b} \). Choose \( k \) to be the largest integer such that \( x = (2 + 3k)n + y \) and \( y \geq 0 \). Then,

\[
\sigma(a)^{b} \leq (2k + 2m) + 2(2y - 1)
\]

**Claim 4.** We can construct any Torified Rational tangle such that it is contained entirely in a twist-box, where each strand will only intersect the box at its exit and entry points, which are on opposite faces of the box.

**Claim 5.** Let \( L \) be a three-box Torified Rational Link. If all of the twist-boxes of \( L \) have supercoils, then we can always attack the skeleton in the twist-boxes by adding \( 2w_2 + 2w_2 \) new sticks.

Figure 8: The box strand number.

**References**