The n-sided double cake is circumscribed by an n-sided right prism, whose basis is the regular n-gon with apothem r, as shown in Figure 2. Thus the volume and surface area of the circumscribing prism are given by

$$V_p = n\text{-gon area} \cdot \text{prism height} = nr^2 \tan\left(\frac{\pi}{n}\right) \cdot 2r = 2nr^3 \tan\left(\frac{\pi}{n}\right) \tag{3}$$

and

$$\begin{aligned} S_p &= 2n\text{-gon area} + n\text{-gon perimeter} \cdot \text{prism height} \\ &= 2nr^2 \tan\left(\frac{\pi}{n}\right) + 2nr \tan\left(\frac{\pi}{n}\right) \cdot 2r \\ &= 6nr^2 \tan\left(\frac{\pi}{n}\right). \end{aligned} \tag{4}$$

Formulas (1)–(4) imply that the ratio $2{:}3 = V_{dc}{:}V_p = S_{dc}{:}S_p$ between volumes and surface areas is valid for these objects for all $n \geq 3$.

Moreover, as $n \to \infty$, the n-sided doublecake tends to a sphere of radius r, while the corresponding prism tends to its circumscribing cylinder. These limit processes furnish alternative methods for computing the volume and surface area of a sphere. Both limits involve the indeterminate form $n \sin(\pi/n)$, with n tending to infinity, which can be resolved using the basic trigonometric limit $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$. Finally, these limit processes generalize the calculus of the area of the circle as a limit of the areas of circumscribed polygons. In fact, this limit occurs in the central cross section of the doublecake.

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Pythagorean Triples with Square and Triangular Sides

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Fermat [**2**] proved there are no Pythagorean triples in which the two smaller numbers (or legs) are both squares. On the other hand, Sierpinski [**3**] proved there are infinitely many in which both legs are consecutive triangular numbers. We begin this note by considering triples with one leg of each type, an example of which is (3, 4, 5) where $3 = t(2)$. (The nth triangular number is $t(n) = n(n+1)/2$.)

The triple (5, 12, 13) does not have our property. Yet if we multiply it by 3, the squarefree part of $12 = 2^2 \cdot 3$, we get (15, 36, 39) where $15 = t(5)$ and $36 = 6^2$. In general, we will take any Pythagorean triple with a square leg and multiply all three numbers by an appropriate square (1 in this example) so that the other leg becomes triangular.

Let (a, b, c) be a Pythagorean triple and let β be the square-free part of b. Then $(\beta a, \beta b, \beta c)$ is a Pythagorean triple with square leg βb. We want to find squares y^2 so that $(y^2\beta a, y^2\beta b, y^2\beta c)$ is a Pythagorean triple and $y^2\beta a$ is a triangular number. That is, we wish to find n so that $y^2\beta a = n(n+1)/2$. Rearranging gives the quadratic equation $0 = n^2 + n - 2y^2\beta a$. Since n is an integer, the discriminant of this quadratic, $1 + 8y^2\beta a$, must be a square, say x^2, which gives a Pell's equation $x^2 - 8\beta a y^2 = 1$.

It is known that $x^2 - dy^2 = 1$ has infinitely many solutions if and only if d is not a square. If there are solutions, the fundamental or smallest one (x_1, y_1) generates all solutions from the rational and irrational parts of the powers $(x_1 + y_1\sqrt{d})^k$ for $k = 2, 3, \ldots$. It follows that $y_2 = 2x_1y_1$ and $y_{k+1} = 2x_1y_k - y_{k-1}$ for $k = 3, 4, \ldots$.

In our problem, $d = 8\beta a$ and we need to verify that d is not square. Suppose that it is. Then the area of the triangle $(\beta a, \beta b, \beta c)$ is $\frac{1}{2}(\beta a)(\beta b) = \frac{1}{16}(8\beta a)(\beta b) = (\frac{d}{16})(\beta b)$, a square. But Fermat proved that the area of a right triangle with integer sides is never square [**2**]. So our assumption that d is square is false. Since $x^2 - 8\beta a y^2 = 1$ has infinitely many solutions, there are infinitely many squares that produce Pythagorean triples with the desired property. We have proved the following theorem.

Theorem. *Every Pythagorean triple has infinitely many multiples with the property that one leg is square and the other leg is triangular.*

For the example $(5, 12, 13)$, $\beta = 3$ and $a = 5$, so $d = 120$. The equation $x^2 - 120y^2 = 1$ has fundamental solution $(11, 1)$. Applying the recursive formula, we find that $y_2 = 22$ and $y_3 = 483$. We get triples $(15 \cdot 22^2, 36 \cdot 22^2, 39 \cdot 22^2)$ and $(15 \cdot 483^2, 36 \cdot 483^2, 39 \cdot 483^2)$ with triangular numbers $15 \cdot 22^2 = 7260 = t(120)$ and $15 \cdot 483^2 = 3499335 = t(2645)$.

Students can work out the second triple in the sequences for $(3, 4, 5)$ and $(4, 3, 5)$. The first gives $(t(24), 20^2, 500)$ while the second gives $(t(24), 15^2, 375)$, showing that the order of the legs is significant.

A special case arises from Pythagorean triples whose legs are consecutive integers. See [**1**] for a description of infinitely many such triples. We will use the parametrization of primitive triples given by $(m^2 - n^2, 2mn, m^2 + n^2)$, where $\gcd(m, n) = 1$ and either m or n is even.

Corollary. *If (a, b, c) is a Pythagorean triple, b is even, and $a - b = \pm 1$, then $(ab/4, b^2/4, bc/4)$ is a Pythagorean triple in which $ab/4$ is a triangular number and $b^2/4$ is a square.*

Proof. Suppose $a = m^2 - n^2$, $b = 2mn$, and $a - b = \pm 1$. Clearly $b^2/4 = m^2n^2$ is square. It suffices to show that $ab/4$ is triangular. We have

$$ab/4 = (m^2 - n^2)(mn/2) = (m - n)(m + n)(m)(n)/2 = (m^2 - mn)(n^2 + mn)/2,$$

where

$$(m^2 - mn) - (n^2 + mn) = m^2 - n^2 - 2mn = a - b = \pm 1. \qquad \blacksquare$$

So far, we have constructed Pythagorean triples of the form (t, s, x), where t, s and x represent numbers which are triangular, are square, and have no restriction, respectively. Similarly, every Pythagorean triple (a, b, c) has infinitely many multiples of the forms (x, t, s) and (x, s, t). Both arguments proceed in the same manner except that we cannot use the Fermat area to reach a contradiction. Assuming the Pell d is square leads to $2bc$ being square. Without loss of generality, assume (a, b, c) is primitive and parametrize. Then by standard arguments, the parameters, m and n, are both square. Finally, the square $2bc = 4mn(m^2 + n^2)$ implies a solution to the unsolvable $x^4 + y^4 = z^2$. Our original (t, s, x) case can also be proved this way with the argument ending at the unsolvable $x^4 - y^4 = z^2$ instead.

Many authors have considered similar problems. For example, Gerry Myerson noticed the triple $(27^2, t(80), t(81))$. Sierpinski found $(t(132), t(143), t(164))$, the only known (t, t, t) example, as well as the infinite family of triples with consecutive triangular legs like $(t(6), t(7), 35)$. And R. P. Burn found many examples of (t, t, x) which are not included in Sierpinski's work such as $(t(8), t(14), 111)$.

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Bernstein's Examples on Independent Events

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In 1946, S.N. Bernstein [**1**, p. 47] gave two examples showing that if the events in a set are *pairwise* independent, they need not be *jointly* independent. In both examples, the sample space has four outcomes, all equally likely. This raises the question of whether there are smaller examples or others of the same size. In this note, we show that the answer to both parts is that there are not. For the sake of simplicity, all of the sample spaces we discuss are assumed to be finite and to have at least two outcomes, with each outcome having positive probability.

We recall the key definitions. Given an experiment, two events A and B are *independent* if $P(A \cap B) = P(A)P(B)$. More generally, events $A_1, A_2, \dots, A_k$ are *jointly independent* if $P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_j})$ for every subset $\{i_1, i_2, \dots, i_j\}$ of $\{1, 2, \dots, k\}$. Obviously, jointly independent sets are pairwise independent.

What Bernstein's examples show is that the converse is not true. For our version of his first example [**1**, p. 47], we consider an urn containing four balls, numbered 110, 101, 011 and 000, from which one ball is drawn at random. For $i = 1, 2, 3$ let A_i be the event of drawing a ball with 1 in the ith position. Thus, the three events are pairwise independent. However, since $A_1 \cap A_2 \cap A_3 = \varnothing$, they are not jointly independent.

In his second example, Bernstein used a tetrahedron with colored faces (one red, one blue, one green, and one with all three colors). We give an equivalent example using the same sample space as in the first example, but with events A_1, A_2, and A_3 where A_i is the event of drawing a ball with a 0 in the ith position, not a 1. Note that each $P(A_i) = \frac{1}{2}$ and each $P(A_i \cap A_j) = \frac{1}{4}$ for $i \neq j$, so the three events are again pairwise

