The Converse of Viviani's Theorem

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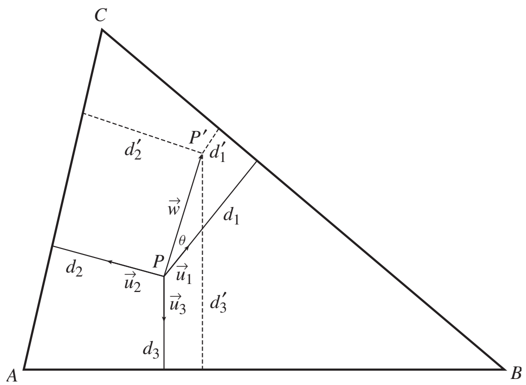
Viviani's Theorem, discovered over 300 years ago, states that inside an equilateral triangle, the sum of the perpendicular distances from a point P to the three sides is independent of the position of P (and so equals the altitude of the triangle). Incidentally, the person for whom this theorem is named is Vicenzo Viviani (1622–1703), a pupil of both Galileo and Torricelli.

The theorem can easily be proved: Let s and h be the side length and the altitude of the equilateral triangle $\triangle ABC$, let P be any point inside the triangle, and let d_1, d_2, and d_3 be the three distances from P to the sides of the triangle. Since $\triangle ABC$ is made up of $\triangle PAB$, $\triangle PBC$, and $\triangle PCA$, it follows that $\frac{1}{2}sh = \frac{1}{2}sd_1 + \frac{1}{2}sd_2 + \frac{1}{2}sd_3$, and so $d_1 + d_2 + d_3 = h$, which proves the theorem.

Samelson [**1**] gave a different proof, one that uses vectors, something we will return to shortly. This motivated us to consider the following question: Does the converse of Viviani's Theorem also hold? That is, if the sum of the distances from a point inside a triangle to the three sides is constant, must the triangle be equilateral? An affirmative answer can be obtained just by considering different points near one vertex of the triangle. However, by using vectors (in the style of Samelson), we can get a stronger result.

Theorem 1. *If, inside $\triangle ABC$, there is a circular region R for which the sum of the distances from a point P in R to the three sides of the triangle is independent of the position of P, then $\triangle ABC$ is equilateral.*



Proof. Let P be a point in R, and let $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ be the unit vectors from P perpendicular to the sides of the triangle (see figure). Our goal is to show that the angle between any two of these vectors is 120°, from which it will follow that each angle of the triangle is 60°.

To this end, we first show that the sum of these vectors, $\vec{u} = \vec{u}_1 + \vec{u}_2 + \vec{u}_3$, is $\vec{0}$. Suppose not. From our hypothesis, it follows that there is a point P' in R so that $\overrightarrow{PP'}$ is parallel to $\vec{u}$. Let $\vec{w}$ denote the vector $\overrightarrow{PP'}$, and let θ be the angle between $\vec{u}_1$ and $\vec{w}$. Further, let d_1, d_2, and d_3 be the distances from P to the sides of $\triangle ABC$, and let d_1', d_2', and d_3' be the corresponding distances from P'. Note that by hypothesis $d_1 + d_2 + d_3 = d_1' + d_2' + d_3'$.

On the one hand,

$$\cos\theta = \frac{d_1 - d_1'}{|\vec{w}|},$$

while on the other hand,

$$\cos\theta = \frac{\vec{u_1} \cdot \vec{w}}{|\vec{w}|}$$

(since $\vec{u_1}$ is a unit vector). Hence, $\vec{u_1} \cdot \vec{w} = d_1 - d_1'$; and by symmetry, $\vec{u_2} \cdot \vec{w} = d_2 - d_2'$ and $\vec{u_3} \cdot \vec{w} = d_3 - d_3'$. It follows from a little algebra that $\vec{u} \cdot \vec{w} = 0$, and since these two vectors are parallel, it must be that $|\vec{u}| = 0$, a contradiction.

From this it follows that, for $i = 1$, 2, and 3, $\vec{u}_i \cdot \left(\vec{u}_1 + \vec{u}_2 + \vec{u}_3\right) = 0$. It is now straightforward to show that $\vec{u}_1 \cdot \vec{u}_2 = \vec{u}_2 \cdot \vec{u}_3 = \vec{u}_3 \cdot \vec{u}_1 = -\frac{1}{2}$. Consequently, the angle between any pair of these vectors is $\frac{2\pi}{3}$, and so $\triangle ABC$ must be equilateral. ■

We note that while the areas method used in the first proof of Viviani's Theorem apparently cannot be used to establish the converse, it can be used to extend the theorem to all regular polygons.

Theorem 2. *The sum of the distances from any point P inside a regular polygon to the sides of the polygon is independent of the location of P.*

The converse of this more general result only holds for triangles however. For example, it is not hard to show that the sum of the distances from any point inside a parallelogram to the sides is independent of the location of that point. In fact, a good student should be able to use the vector method to prove the following result:

Theorem 3. *Inside a quadrilateral, the sum of the perpendicular distances from a point P to the four sides is independent of the location of P if and only if the quadrilateral is a parallelogram.*

There is also an extension of Viviani's Theorem to three dimensions:

Theorem 4. *The sum of the distances from any point P inside a regular polyhedron to the sides of the polyhedron is independent of the location of P.*

We note that the converse of this result does not hold, even for tetrahedra (this can easily be shown using the vector method).

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References

1. H. Samelson, Proof Without Words: Viviani's Theorem with Vectors, *Math Mag.* 76 (2003) 225.