The Fresnel Integrals Revisited

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Many articles [**2**–**5**] have been devoted to determining the values of the Fresnel integrals:

$$F_c = \int_0^\infty \cos x^2\,dx \quad \text{and} \quad F_s = \int_0^\infty \sin x^2\,dx.$$

For example, Flanders [**2**] considers

$$F_c(t) = \int_0^\infty e^{-tx^2} \cos x^2\,dx \quad \text{and} \quad F_s(t) = \int_0^\infty e^{-tx^2} \sin x^2\,dx \tag{1}$$

and shows that they satisfy the functional equations

$$F_c^2(t) - F_s^2(t) = \frac{\pi t}{4(1+t^2)}, \quad 2F_c(t)F_s(t) = \frac{\pi}{4(1+t^2)}.$$

He then solves these simultaneous quadratic equations to find the values of $F_c(t)$ and $F_s(t)$. The cleverness of his method resides in the introduction of polar coordinates, as is usually done to evaluate $\int_0^\infty e^{-x^2}\,dx$.

The present note offers a simpler method which does not use the double integral, nor the system of quadratic equations. The main ingredients of the method are the consideration of some related derivatives and linear differential equations. The method applies to a number of pairs of integrals.

The method

In analogy with the function pairs (1), for $t > 0$, $p \in \mathbb{R}$, we introduce

$$F_c(p) = \int_0^\infty e^{-tx^2} \cos(px^2)\,dx, \quad F_s(p) = \int_0^\infty e^{-tx^2} \sin(px^2)\,dx.$$

We will prove that

$$F_c'(p) = -\frac{pF_c(p) + tF_s(p)}{2(t^2+p^2)}, \quad F_s'(p) = \frac{tF_c(p) - pF_s(p)}{2(t^2+p^2)}, \tag{2}$$

and if $Z(p) = F_c(p) + i\, F_s(p)$, that

$$Z(p) = \frac{\sqrt{\pi}}{2} \frac{\sqrt{t+pi}}{\sqrt{t^2+p^2}}. \tag{3}$$

Since

$$\sqrt{t+pi} = \sqrt{\frac{t+\sqrt{t^2+p^2}}{2}} + \sqrt{\frac{-t+\sqrt{t^2+p^2}}{2}}\, i,$$

upon establishing (3), as an immediate consequence we find

$$F_c(p) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{\frac{t+\sqrt{t^2+p^2}}{t^2+p^2}},$$

$$F_s(p) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\sqrt{\frac{-t+\sqrt{t^2+p^2}}{t^2+p^2}}.$$

The dominated convergence theorem [**1**] asserts that $F_c(p)$ and $F_s(p)$ both converge uniformly for all $t > 0$. Hence F_c and F_s are continuous functions of t. In particular, setting $p = 1$ and letting $t \to 0^+$, we obtain the Fresnel integrals

$$\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

as desired.

To prove (2), for any $p \in \mathbb{R}$ and $t \geq t_0 > 0$, since

$$|x^2 e^{-tx^2} \cos(px^2)| \leq x^2 e^{-t_0 x^2}, \quad |x^2 e^{-tx^2} \sin(px^2)| \leq x^2 e^{-t_0 x^2}$$

and $\int_0^\infty x^2 e^{-t_0 x^2}\, dx$ converges, the integrals

$$\int_0^\infty x^2 e^{-tx^2} \cos(px^2)\, dx \quad \text{and} \quad \int_0^\infty x^2 e^{-tx^2} \sin(px^2)\, dx$$

converge uniformly for any p. By using the Leibniz rule, differentiating under the integral sign gives

$$F_c'(p) = -\int_0^\infty x^2 e^{-tx^2} \sin(px^2)\, dx, \quad F_s'(p) = \int_0^\infty x^2 e^{-tx^2} \cos(px^2)\, dx.$$

Next, integrating by parts leads to

$$\begin{aligned} F_c'(p) &= \frac{1}{2t}\int_0^\infty x \sin(px^2)\, d(e^{-tx^2}) \\ &= \frac{1}{2t}\left(-\int_0^\infty e^{-tx^2} \sin(px^2)\, dx - 2p\int_0^\infty x^2 e^{-tx^2} \cos(px^2)\, dx\right) \\ &= -\frac{1}{2t} F_s(p) - \frac{p}{t} F_s'(p). \end{aligned}$$

Similarly,

$$F_s'(p) = \frac{1}{2t} F_c(p) + \frac{p}{t} F_c'(p).$$

Finally, solving for $F_c'(p)$ and $F_s'(p)$ yields (2) as expected.

We prove (3) by observing:

$$\frac{d}{dp} Z(p) = \frac{-p + ti}{2(t^2 + p^2)} Z(p) = \frac{i}{2} \frac{Z(p)}{t - pi}. \tag{4}$$

To solve this equation, instead of using separation of variables, which will involve the complex logarithm, direct calculation shows that (4) is equivalent to

$$\frac{d}{dp}\left(Z(p) \cdot \sqrt{t - pi}\right) = 0.$$

Thus,

$$Z(p) \cdot \sqrt{t - pi} = \text{const.}$$

Notice that

$$Z(0) = F_c(0) = \int_0^\infty e^{-tx^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}.$$

It follows that

$$Z(p) = \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{t - pi}} = \frac{\sqrt{\pi}}{2} \frac{\sqrt{t + pi}}{\sqrt{t^2 + p^2}}$$

as claimed.

Further applications

It is interesting to see that this technique is also strong enough to capture the parametric integrals:

$$G_c(p) = \int_0^\infty e^{-x^2} \cos\left(\frac{p^2}{x^2}\right) dx, \quad G_s(p) = \int_0^\infty e^{-x^2} \sin\left(\frac{p^2}{x^2}\right) dx.$$

None of them can be evaluated by the double integral via polar coordinates. By pursuing the same line of reasoning as used above, we have

$$G_c''(p) = 4G_s(p), \quad G_s''(p) = -4G_c(p).$$

Furthermore, let $W(p) = G_c(p) + G_s(p)\, i$. Then $W''(p) = -4i\, W(p)$. This yields, subject to the boundedness of $W(p)$ and $W(0) = \sqrt{\pi}/2$,

$$W(p) = \frac{\sqrt{\pi}}{2} e^{-\sqrt{2}p} \cos(\sqrt{2}p) + i\, \frac{\sqrt{\pi}}{2} e^{-\sqrt{2}p} \sin(\sqrt{2}p).$$

Thus,

$$G_c(p) = \frac{\sqrt{\pi}}{2} e^{-\sqrt{2}p} \cos(\sqrt{2}p), \quad G_s(p) = \frac{\sqrt{\pi}}{2} e^{-\sqrt{2}p} \sin(\sqrt{2}p).$$

For interested readers, establishing the following formulas shows the wider applicability of our method.

$$\int_0^\infty \sin(a^2x^2)\cos\left(\frac{p^2}{x^2}\right)dx = \frac{1}{4a}\sqrt{\frac{\pi}{2}}\left(\sin 2ap + \cos 2ap + e^{-2ap}\right),$$

$$\int_0^\infty \cos(a^2x^2)\cos\left(\frac{p^2}{x^2}\right)dx = \frac{1}{4a}\sqrt{\frac{\pi}{2}}\left(\cos 2ap - \sin 2ap + e^{-2ap}\right),$$

$$\int_0^\infty e^{-ax^2}x^{2p-1}\cos(bx^2)\,dx = \frac{\Gamma(p)}{2(a^2+b^2)^{p/2}}\cos(p\theta_0),$$

$$\int_0^\infty e^{-ax^2}x^{2p-1}\sin(bx^2)\,dx = \frac{\Gamma(p)}{2(a^2+b^2)^{p/2}}\sin(p\theta_0),$$

where $a > 0$, $p > 0$ and $\theta_0 = \arctan(b/a)$.

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References

1. T. M. Apostol, *Mathematics Analysis*, 2nd ed., Addison-Wesley, 1974.
2. H. Flanders, On the Fresnel integrals, *Amer. Math. Monthly* **89**(1982) 264–266.
3. I. E. Leonard, More on Fresnel integrals, *Amer. Math. Monthly* **95** (1988) 431–433.
4. R. Weinstock, Elementary evaluations of the Fresnel integrals, *Amer. Math. Monthly* **97**(1990) 39–42.
5. J. Yzeren, Moivre's and Fresnel's integrals by simple integration, *Amer. Math. Monthly* **86** (1979) 691–693.