CLASSROOM CAPSULES

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Classroom Capsules are short (1–3 page) notes that contain new mathematical insights on a topic from undergraduate mathematics, preferably something that can be directly introduced into a college classroom as an effective teaching strategy or tool. Classroom Capsules should be prepared according to the guidelines on the inside front cover and sent to any of the above editors.

Derivative Sign Patterns

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The function e^x has many unusual properties, including this one: it and all its derivatives are positive for every single x.

This generalizes to e^{cx} as long as $c > 0$; on the other hand, e^{-x} and its derivatives alternate between being always positive and always negative.

How can we articulate this behavior, and can we generalize it beyond the exponential functions?

Sign patterns A *sign pattern* is a sequence (a_n) where $a_n \in \{\pm 1\}$ for $n = 0, 1, 2, 3, \ldots$. An infinitely differentiable function $f\colon A \to \mathbb{R}$ on an open set $A \subseteq \mathbb{R}$ has the *derivative sign pattern* (a_n) if for all $x \in A$ and all $n = 0, 1, 2, 3, \ldots$ $\operatorname{sign}\left(f^{(n)}(x)\right) = a_n$. (Here the signum function $\operatorname{sign}(y)$ is 1, if y is positive, -1, if y is negative, and 0 if y is 0.) In other words, such functions and their derivatives never change sign on their domain, and the derivative sign pattern keeps track of which side of 0 they are on.

Polynomials never have sign patterns, since if you take enough derivatives you are left with the zero function. It is also commonplace for a function or one of its derivatives to change signs somewhere in its domain; such functions do not have sign patterns, either.

In our examples above, e^x has sign pattern $(1, 1, 1, 1, \ldots)$, while e^{-x} has sign pattern $(1, -1, 1, -1, \ldots)$. Negating a function with a derivative sign pattern negates the signs in that pattern: $-e^x$ has sign pattern $(-1, -1, -1, -1, \ldots)$, and $-e^{-x}$ has sign pattern $(-1, 1, -1, 1, \ldots)$. These pairs of functions are also related in that negating x as an input to the function flips the signs for odd n.

So far, we've seen four derivative sign patterns: $(1, 1, 1, 1, \ldots)$, $(1, -1, 1, -1, \ldots)$, $(-1, -1, -1, -1, \ldots)$, and $(-1, 1, -1, 1, \ldots)$.

Is any other sign pattern possible? If A is the entire real line, the answer is no. To show this, we need the following theorem:

Theorem. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be twice differentiable. If $f(x) > 0$ for all x, then $f''(x_0) \geq 0$ for some $x_0 \in \mathbb{R}$.*

In other words, it is impossible for a twice-differentiable function defined on all reals to be both positive and concave down everywhere.

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Proof. If f' is identically 0, then f'' is as well and we are done. Assume now that f' is not identically 0.

If the conclusion were false, then $f''(x) < 0$ for all x. Since f' is not identically 0, there is a real number a such that $f'(a) \neq 0$. By Taylor's Theorem,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)(x-a)^2}{2},$$

for some c between a and x. Let $x_0 = a - f(a)/f'(a)$, where the tangent line to $f(x)$ at $x = a$ strikes the x-axis. Recalling that, by assumption, $f''(c) < 0$ we have

$$f(x_0) = \frac{f''(c)(x_0-a)^2}{2} < 0,$$

a contradiction. Therefore f'' is not always negative. ■

The proof depends on the fact that a curve which is concave downward must lie below its tangent line; since any non-horizontal tangent line crosses the x-axis, the curve must also cross the x-axis. This argument fails when the function's domain is a proper subset of the real number line.

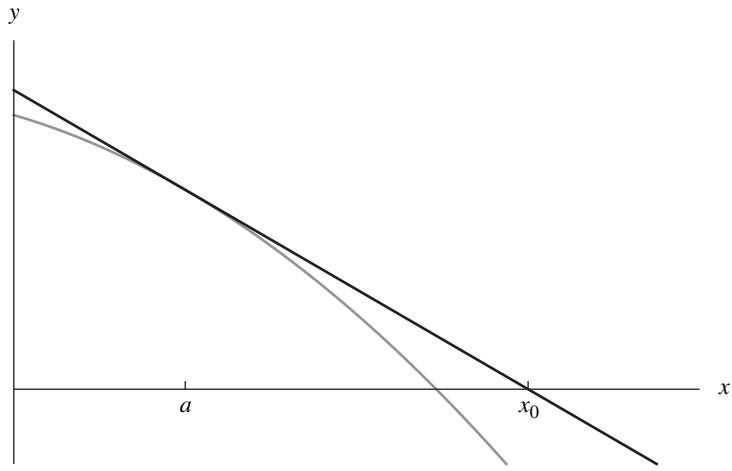

Figure 1.

Corollary. *If $A = \mathbb{R}$, the only derivative sign patterns are* $(1, 1, 1, 1, \ldots)$, $(1, -1, 1, -1, \ldots)$, $(-1, -1, -1, -1, \ldots)$, *and* $(-1, 1, -1, 1, \ldots)$.

Proof. Note that by negating the function we have that if f is always negative, f'' can't be always positive.

If (a_n) is a derivative sign pattern, then $a_n = a_{n+2}$ for all n. The derivative sign pattern is completely determined by a_0 and a_1, yielding the four patterns listed above. ■

Restricting the domain If we want derivative sign patterns other than the four we have listed, we have to restrict the domain in some way. For a bounded domain that is relatively simple, we start with an open interval.

If the domain of f is a bounded open interval (a, b), then we can always look instead at $g(x) = f(a + (b - a)x)$ on $(0, 1)$. f' and g' will differ by a factor of $b - a > 0$ and higher derivatives will also differ by a positive multiplicative constant. If f has a derivative sign pattern, then g will have the same derivative sign pattern. We therefore focus on the domain $(0, 1)$.

Theorem. *For any sign pattern (a_n) there exists an infinitely differentiable function $f\colon (0, 1) \to \mathbb{R}$ with derivative sign pattern (a_n).*

Proof. Define $f\colon (0, 1) \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{2^n n!},$$

a modification of the Taylor series for $e^{x/2}$, where we revise the sign of the terms to match the given sign pattern.

On $(0, 1)$, each term in the series is bounded by $1/2^n$. By the Weierstrass M-test, it converges uniformly. Therefore $f(x)$ is defined and infinitely differentiable. If we differentiate $f(x)$ repeatedly term by term,

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=0}^{\infty} \frac{a_n n(n-1)\cdots(n-k+1)x^{n-k}}{2^n n!} \\ &= \sum_{n=0}^{\infty} \frac{a_{n+k} x^n}{2^{n+k} n!} \\ &= \frac{a_k}{2^k} + \sum_{n=1}^{\infty} \frac{a_{n+k} x^n}{2^{n+k} n!}. \end{aligned}$$

It follows that $f^{(k)}(x)$ has the same sign as $a_k/2^k$, i.e., the same sign as a_k, if the magnitude of the sum on the right-hand side is less than $1/2^k$, the magnitude of $a_k/2^k$. We find that

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \frac{a_{n+k} x^n}{2^{n+k} n!} \right| &\le \sum_{n=1}^{\infty} \left| \frac{a_{n+k} x^n}{2^{n+k} n!} \right| \\ &\le \sum_{n=1}^{\infty} \frac{1}{2^{n+k} n!} \\ &= \frac{1}{2^k} \left(e^{1/2} - 1 \right) \\ &< \frac{1}{2^k} \\ &= \left| \frac{a_k}{2^k} \right|, \end{aligned}$$

therefore $f(x)$ has derivative sign pattern (a_n). ■

Further exploration What derivative sign patterns are there for other domains in $\mathbb{R}$, such as $(0, \infty)$? What if we extend the definition of derivative sign patterns to functions of more than one variable and include partial derivatives? These are just a few of the open questions worth exploration.

Summary. Analysis of the patterns of signs of infinitely differentiable real functions shows that only four patterns are possible if the function is required to exhibit the pattern at all points in its domain *and* that domain is the set of all real numbers. On the other hand *all* patterns are possible if the domain is a bounded open interval.

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Limit Interchange and L'Hôpital's Rule

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Students eventually learn the importance and unifying power of interchanging limits with other operations, like summation and integration, through advanced results like Lebesgue's Dominated Convergence Theorem and Fubini's Theorem. Even in an introductory analysis course, limit interchange shows up in connection with uniform convergence. Can the concept of limit interchange be introduced still earlier—in a first-year calculus course? L'Hôpital's rule provides an opportunity.

A geometric sum This example, though simple, is characteristic. The goal is to add a finite number of 1's:

$$\begin{aligned} 1 + 1 + \cdots + 1 \ (n \text{ terms}) &= 1 + \lim_{r\to 1} r + \lim_{r\to 1} r^2 + \cdots + \lim_{r\to 1} r^{n-1} \ (n \text{ terms}) \\ &= \lim_{r\to 1}\left(1 + r + \cdots + r^{n-1}\right) = \lim_{r\to 1}\left(\frac{r^n - 1}{r - 1}\right) \qquad (1) \\ &= \lim_{r\to 1}\left(\frac{nr^{n-1}}{1}\right) = n \end{aligned}$$

Passage from the first line to the second is by interchanging the sum with a limit; passage to the last line is by l'Hôpital's Rule. The final result is ridiculously obvious—comical, but also crucial, because it proves the validity of the limit interchange.

Limits under the integral sign Integrals containing a parameter are a natural domain of application for l'Hôpital's Rule. Here is one example:

$$\int \frac{dt}{t^2 + a^2} = \begin{cases} \dfrac{1}{a}\arctan\left(\dfrac{t}{a}\right) + C & a > 0 \\ -\dfrac{1}{t} + C & a = 0, \end{cases}$$

where t must be non-negative. Is it permissible to pass from $a > 0$ to $a = 0$ by a limit under the integral sign? The convergence here (as a approaches 0) is uniform,

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