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Infinite Series with Binomial Coefficients

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As a result of Apery's 1978 announcement [4] of the irrationality of

$$\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{n^3}$$

there arose interest in evaluating series of the form

$$S(k)=\sum_{n=1}^{\infty}\frac{1}{n^k\binom{2n}{n}}$$

and

$$T(k)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^k\binom{2n}{n}}.$$

The special cases of $S(k)$, for $k=0,1,2,4$, and $T(k)$, for $k=0,1,2,3$ appear to have been known for some time [5]. More recently, evaluations of $S(3)$ and $S(5)$ have appeared [6].

In this note we give an alternative method for evaluating some of these sums, as well as some other series involving binomial coefficients in the denominator. We also show how the problem becomes much easier if we alter it by adding some terms to the series.

Recall the standard identity [1]

$$\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}=B(s,t) \tag{1}$$

where

$$\Gamma(s)=\int_0^{\infty}e^{-u}u^{s-1}\,du,\qquad s>0,$$

and

$$B(s,t) = \int_0^1 u^{s-1}(1-u)^{t-1}\,du, \qquad s,t > 0,$$

are the well-known gamma and beta functions respectively. We illustrate our method by means of an example.

$$\sum_{n=1}^{\infty} \frac{1}{n\binom{2n}{n}} = \sum_{n=1}^{\infty} \frac{1}{n}\frac{n!n!}{(2n)!} = \sum_{n=1}^{\infty} \frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)}$$

$$= \sum_{n=1}^{\infty} B(n,n+1) = \sum_{n=1}^{\infty} \int_0^1 u^n(1-u)^{n-1}\,du = \int_0^1 u \sum_{n=1}^{\infty} (u(1-u))^{n-1}\,du$$

$$= \int_0^1 \frac{u}{u^2-u+1}\,du = \frac{\pi}{3\sqrt{3}}.$$

$\sum_{n=1}^{\infty} 1/\binom{2n}{n}$ is handled in a similar fashion. For the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}},$$

we arrive at the integral

$$-\int_0^1 \frac{\log(u^2-u+1)}{u}\,du, \tag{2}$$

which appears in the study of the dilogarithm. Consulting the standard reference [3], we find that the integral (2) equals $2\mathrm{Li}_2(1,\pi/3) = \pi^2/18$.

A standard method for evaluating these series (see [2]) begins with the expansion

$$\frac{2x\sin^{-1}(x)}{\sqrt{1-x^2}} = \sum_{m=1}^{\infty} \frac{(2x)^{2m}}{m\binom{2m}{m}}, \qquad |x| < 1.$$

This formula can easily be derived from (1) by the same method we used to evaluate $\sum_{n=1}^{\infty} 1/n\binom{2n}{n}$.

A typical series that this method can evaluate and that we have not seen previously is

$$\sum_{n=1}^{\infty} \frac{1}{n\binom{3n}{n}}.$$

This series leads to the integral

$$\int_0^1 \frac{u^2}{u^3-u^2+1}\,du,$$

which can be evaluated explicitly with partial fractions. The final result is given in terms of the real zero of u^3-u^2+1, but the expression is unenlightening and too complicated to reproduce.

Our second remark is that if one considers not just series with the central binomial coefficient $\binom{2n}{n}$ in the denominator but series involving all possible entries $\binom{n}{k}$ in

Pascal's triangle, the problem becomes much easier.

To be specific, consider the series

$$\sum_{k=2}^{\infty} \sum_{n=k}^{\infty} \frac{1}{k^m \binom{n}{k}}, \qquad m \geqslant 2. \tag{3}$$

We neglect the terms involving $\binom{n}{0}$ and $\binom{n}{1}$ to insure convergence. Equation (1) and the above method give an easy proof that for $k \geqslant 2$,

$$\sum_{n=k}^{\infty} \frac{1}{\binom{n}{k}} = \frac{k}{k-1}. \tag{4}$$

If $m = 2$, the sum (3) therefore equals

$$\sum_{k=2}^{\infty} \frac{1}{k^2} \frac{k}{k-1} = 1.$$

If $m \geqslant 3$, we use partial fractions again to see that (3) equals

$$1 + \sum_{j=2}^{m-1} (1 - \zeta(j)),$$

where $\zeta(j) = \sum_{n=1}^{\infty} 1/n^j$. For $m = 0$ and $m = 1$ we must exclude more terms to have the series converge. The analogous results in these cases are

$$\sum_{k=2}^{\infty} \sum_{n=k+1}^{\infty} \frac{1}{k \binom{n}{k}} = 1$$

and

$$\sum_{k=2}^{\infty} \sum_{n=k+2}^{\infty} \frac{1}{\binom{n}{k}} = \frac{3}{2}.$$

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