A Curious Sequence

HERB R. BAILEY
ROGER G. LAUTZENHEISER
Rose-Hulman Institute of Technology
Terre Haute, IN 47803

We call the finite sequence $a_0, a_1, a_2, \dots, a_n$ *curious* if a_i is the number of i's in the sequence for each $i = 0, 1, \dots, n$. For example, $1, 2, 1, 0$ is a curious sequence with $n = 3$. The problem is to find all such sequences for any n.

This is a good problem for students at almost any level since some thinking is required just to understand the definition. Solutions can be found for small values of n by simple experimentation. For larger n, good students can be challenged by writing a computer program to generate solutions. In addition to finding solutions for large values of n, very good students might be able to prove that these solutions are unique. The problem for $n = 7$ was in the 1978–79 Scottish Mathematical Challenge Examination and the problem for $n = 10$ was in the 1987–88 Wisconsin Mathematics Science and Engineering Talent Search Examination. This sequence also appears as one of a number of interesting mathematical investigations discussed in [1, pages 23–34].

Before giving a general solution, we note two properties of curious sequences. If $s = \sum_{i=0}^{n} a_i$ and $w = \sum_{i=0}^{n} i a_i$ (w for weighted sum), then both s and w equal $n + 1$. That is, s and w are just two different ways of counting how many elements there are in the sequence and thus they both equal $n + 1$.

THEOREM. *For each $n \geqslant 6$, the sequence*

$$a_0 = n - 3,\ a_1 = 2,\ a_2 = 1,\ a_{n-3} = 1, \text{ and } a_i = 0 \text{ otherwise}$$

is a curious sequence and it is unique.

Proof. Since the sequence satisfies the definition of a curious sequence, we need only prove uniqueness. We first observe that a_0 cannot be either 0 or 1. If $a_0 = 0$ then we have an immediate contradiction. If $a_0 = 1$, then all but one of the remaining a_i's are not equal to 0. Thus at least two of the values a_{n-2}, a_{n-1}, and a_n must be non-zero. However, this implies that w is greater than $n + 1$.

We next show that $a_0 = n - 3$ and $a_1 = 2$. Let $a_0 = j$ (by the above argument, $j \geqslant 2$) and $a_1 = k$. It follows that $a_j \geqslant 1$. Let $a_j = l$. Since $s = n + 1$, the sum of the remaining a_i's (other than a_0, a_1, and a_j) must be $n + 1 - j - k - l$. Since the *subscript* of each of these remaining a_i's must be at least 2, it follows that

$$w = \sum_{i=0}^{n} i a_i \geqslant (0)(j) + (1)(k) + (j)(l) + 2(n + 1 - j - k - l)$$

or, collecting terms,

$$w \geqslant 2n + 2 - k - 2j + l(j - 2).$$

Since $w = n + 1$, the above inequality reduces to

$$k + 2j - l(j - 2) \geqslant n + 1. \tag{1}$$

Since $l \geqslant 1$, the above inequality becomes

$$k + 2j - j + 2 \geqslant n + 1$$

or

$$k + j \geqslant n - 1. \tag{2}$$

We now show that $k \geqslant 2$ by noting that if $k = a_1 = 0$ or 1, then $a_i \neq 1$ for all $i \neq 1$. In particular $a_j = l \neq 1$ and thus $l \geqslant 2$. But for $k \leqslant 1$ and $l \geqslant 2$, inequality (1) reduces to

$$1 + 2j - 2(j - 2) \geqslant n + 1 \text{ or } n \leqslant 4,$$

which contradicts $n \geqslant 6$ and thus

$$k \geqslant 2. \tag{3}$$

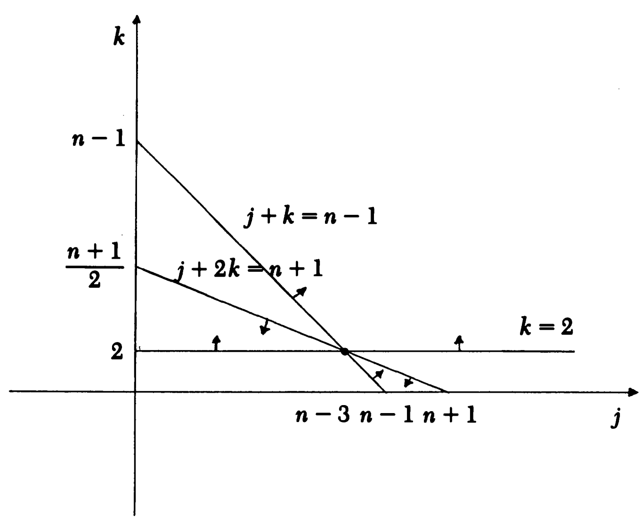
Since $a_0 = j$ and $a_1 = k \geqslant 2$, the k ones in the sequence occur beyond a_1. Therefore,

$$n + 1 = s = \sum_{i=0}^{n} a_i = a_0 + a_1 + \sum_{i=2}^{n} a_i \geqslant j + k + k$$

or

$$n + 1 \geqslant j + 2k. \tag{4}$$

Inequalities (2), (3), and (4) are shown in the following graph and it is seen that the only solution is $a_0 = j = n - 3$ and $a_1 = k = 2$. Since $a_1 = 2$, it follows that a_2 and a_{n-3} are equal to 1 and, therefore, the sequence given in the statement of the theorem is unique.



We leave the solution to the problem for $n \leqslant 5$ to the reader. We note, however, that for some of these n's, there are no curious sequences and for others there are more than one.

REFERENCE

1. A. Gardiner, *Discovering Mathematics—The Art of Investigation*, Oxford University Press, New York, 1987.