Counting Arrangements of 1's and −1's

D. F. BAILEY
Trinity University
San Antonio, TX 78212

It is well known that the nth Catalan number counts the number of sequences with non-negative partial sums that can be formed from n 1's and n −1's. (See [1].) In this paper we derive a formula for the number of such sequences formed from n 1's and k −1's. In the process we produce a non-standard proof that the nth Catalan number is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Because the numbers we define can be evaluated in a manner similar to the binomial coefficients, we use a symbolism reminiscent of the standard notation for the binomial coefficients.

Definition. Let $\left\{ n \atop k \right\}$ denote the number of arrangements of n 1's and k −1's $a_1 a_2 \dots a_{n+k}$ so that

$$a_1 + a_2 + \cdots + a_i \geq 0 \text{ for all } 1 \leq i \leq n+k.$$

We first establish the following.

LEMMA 1.

(i) $\left\{ n \atop 0 \right\} = 1$ *for* $n \geq 0$.

(ii) $\left\{ n \atop 1 \right\} = n$ *for* $n \geq 1$.

(iii) $\left\{ n+1 \atop k \right\} = \left\{ n+1 \atop k-1 \right\} + \left\{ n \atop k \right\}$ *for* $1 < k < n+1$.

(iv) $\left\{ n+1 \atop n+1 \right\} = \left\{ n+1 \atop n \right\}$ *for* $n \geq 1$.

Proof. Statement (i) is obvious and (ii) is nearly so. To establish (ii) note that an arrangement of n 1's and a single −1 occupies $n+1$ positions. The −1 may occupy any of these positions save the first. Thus there are n such arrangements.

For (iii) imagine an arrangement of $n+1$ 1's and k −1's with $k \geq 2$. The last element (that is a_{n+1+k}) is either a 1 or a −1. If it is a 1 then the remaining positions are filled with n 1's and k −1's. Hence these positions can be filled in $\left\{ n \atop k \right\}$ ways. If, on the other hand, the last element is a −1, the remaining positions are filled with $n+1$ 1's and $k-1$ −1's. This can be done in $\left\{ n+1 \atop k-1 \right\}$ ways. Thus (iii) is established.

Finally, to establish (iv) consider an arrangement of $n+1$ 1's and $n+1$ −1's. Note that the last element in the arrangement must be a −1. Indeed if this element is a 1 then since

$$a_1 + a_2 + \cdots + a_{2n+1} + a_{2n+2} = 0$$

it must be the case that

$$a_1 + a_2 + \cdots + a_{2n+1} = -1 < 0$$

contrary to our requirement. Hence the first $2n + 1$ positions are filled with $n + 1$ 1's and n -1's. Thus the assertion of (iv) is valid.

Using the results of Lemma 1, we can construct a triangular array containing the numbers $\left\{ {n \atop k} \right\}$ in somewhat the same manner as the Pascal triangle is constructed. In particular, the nth row and kth column will contain $\left\{ {n \atop k} \right\}$. By part (i) of Lemma 1 each row begins with a 1, in the 0th column. Then the next entry (the entry in column 1) in the nth row will be n for $n \geq 1$, according to part (ii). After that (by (iii)) the entry in a position is determined by adding the element directly above the position and immediately to the left of the position until there are no elements directly above. (That is until we reach the nth column of row n.) The final entry of a row is obtained, by (iv), by repeating the element immediately to the left.

Here is the array through row 8.

$k=$	0	1	2	3	4	5	6	7	8
$n=0$	1								
$n=1$	1	1							
$n=2$	1	2	2						
$n=3$	1	3	5	5					
$n=4$	1	4	9	14	14				
$n=5$	1	5	14	28	42	42			
$n=6$	1	6	20	48	90	132	132		
$n=7$	1	7	27	75	165	297	429	429	
$n=8$	1	8	35	110	275	572	1001	1430	1430

We can, however, produce a closed form for $\left\{ {n \atop k} \right\}$. To this end we require three lemmas.

LEMMA 2. *For* $2 \leq k \leq n$ *we have*

$$\left\{ {n \atop k} \right\} = \sum_{i=k}^{n} \left\{ {i \atop k-1} \right\}.$$

Proof. For $n = 2$ our assertion is simply that $\left\{ {2 \atop 2} \right\} = \left\{ {2 \atop 1} \right\}$, which is obviously valid. Therefore we assume the result for some fixed $n \geq 2$ and consider $\left\{ {n+1 \atop k} \right\}$. If $k \leq n$, the assertion follows easily by Lemma 1 and the inductive hypothesis.

If, on the other hand, $k = n + 1$ we have

$$\left\{ {n+1 \atop n+1} \right\} = \left\{ {n+1 \atop n} \right\} = \sum_{i=n+1}^{n+1} \left\{ {i \atop n} \right\}$$

and again we have the desired result.

An easy argument using (ii) and Lemma 2 proves:

LEMMA 3. *If* $n \geq 2$ *then*

$$\left\{ {n \atop 2} \right\} = \frac{(n-1)(n+2)}{2}.$$

LEMMA 4.

$$\sum_{i=k+1}^{n} (i+1-k)(i+2)(i+3)\dots(i+k)$$
$$= \frac{1}{k+1}(n-k)(n+2)(n+3)\dots(n+1+k).$$

Proof. For $n = k+1$ we have the assertion that

$$(k+2-k)(k+1+2)(k+1+3)\dots(k+1+k)$$
$$= \frac{1}{k+1}(k+1-k)(k+1+2)\dots(k+1+k)(k+1+k+1),$$

which reduces to

$$2 = \frac{1}{k+1}(2k+2).$$

The inductive step follows easily to complete the proof.

We are now ready to derive a closed form for $\left\{ {n \atop k} \right\}$.

THEOREM. *For* $n \geq k \geq 2$

$$\left\{ {n \atop k} \right\} = \frac{(n+1-k)(n+2)(n+3)\dots(n+k)}{k!}.$$

Proof. From Lemma 3 we have the result when $k = 2$. So we assume the result for some $k \geq 2$ and consider $\left\{ {n \atop k+1} \right\}$. From Lemma 2 and the inductive hypothesis we have

$$\left\{ {n \atop k+1} \right\} = \sum_{i=k+1}^{n} \left\{ {i \atop k} \right\} = \sum_{i=k+1}^{n} \frac{(i+1-k)(i+2)(i+3)\dots(i+k)}{k!}.$$

By Lemma 4 this last summation is

$$\frac{1}{k!}\frac{1}{k+1}(n-k)(n+2)(n+3)\dots(n+1+k)$$

or equivalently

$$\frac{1}{(k+1)!}(n+1-(k+1))(n+2)(n+3)\dots(n+(k+1)),$$

completing the induction.

As an aside we note that we can now complete what must be the longest possible derivation of the closed form for the nth Catalan number, C_n.

COROLLARY.

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Set $k = n$ in the formula for $\left\{ n \atop k \right\}$.

REFERENCE

1. Richard A. Brualdi, *Introductory Combinatorics*, 2nd edition, Elsevier Science Publishing Co., Inc., New York, NY, 1992.

The Fermat Point of a Triangle

P. G. SPAIN
University of Glasgow
Glasgow G12 8QW, Scotland

The problem Fermat posed the problem of minimizing the sum of the distances from a point to the vertices of a triangle. It was solved by Torricelli, Cavalieri, and others **[3]**; see also **[1]**, where it is referred to as "Steiner's Problem."

Solution The solution is that there is precisely one point at which the minimum is attained, called the *Fermat point*. It is the unique interior point F of ΔABC for which FA, FB, FC meet at equal angles, if such a point exists; otherwise it is the vertex of the largest angle.

There are several ways to demonstrate this, most of them in terms of classical synthetic geometry. A recent article in this journal **[2]** provides an interesting discussion and a solution using advanced calculus.

I would like to present a simple proof, self-contained, based on the following elementary lemma concerning a trio of unit vectors summing to zero.

LEMMA. *Suppose that u, v, w, are unit vectors such that*

$$u + v + w = 0.$$

Then the angles between the vectors u, v, w are all equal (*to* 120°).

Proof. The vectors u, v, w must form the sides of an equilateral triangle.

Construction It is easy to identify the point F for which FA, FB, FC meet at equal angles, when there is one, i.e., when all the angles of the triangle are less than 120°, by the methods of elementary geometry.

Let ABC be a triangle all of whose angles are less than 120°. Then the loci of interior points R such that

$$\angle BRC = 120^\circ, \quad \angle CRA = 120^\circ, \quad \angle ARB = 120^\circ \tag{1}$$

are circular arcs that intersect at the required point F. This construction shows that there can be *only one* such point. It is equally clear that no such point can exist in a triangle when any angle is 120° or more.