Euler-Cauchy Using Undetermined Coefficients

Doreen De Leon (doreendl@csufresno.edu), Department of Mathematics, California State University–Fresno, Fresno CA 93740-8001 doi:10.4169/074683410X488728

The Euler-Cauchy equation is often one of the first higher order differential equations with variable coefficients introduced in an undergraduate differential equations course. Putting a nonhomogeneous Euler-Cauchy equation on an exam in such a course, I was surprised when some of my students decided to apply the method of undetermined coefficients, which is guaranteed to work only for constant-coefficient equations, and obtained the correct answer! It turns out that we can find a particular solution to this equation using a substitution similar to the standard method of undetermined coefficients, if the right-hand side function is of a certain type, without using variation of parameters or transforming the equation to a constant-coefficient equation and then applying undetermined coefficients.

Such a solution is possible because of the fact, mentioned in many differential equations textbooks, that the Euler-Cauchy equation may be transformed by a change of variables into a constant-coefficient equation by simply defining \( t = e^z \), if we assume \( t > 0 \). Thus, if the right-hand side function \( f(t) \) is a monomial, then \( f(e^z) \) is an exponential function; or if the right-hand side function \( f(t) \) is the product of a monomial and a nonnegative integer power of \( \ln(t) \), then \( f(e^z) \) is the product of a monomial and an exponential function. And, since the new equation is a constant-coefficient equation, the method of undetermined coefficients can be applied, prescribing a solution that is an exponential function, in the first case, and the product of a polynomial and an exponential function in the second case. This leads to a method of undetermined coefficients for the original equation.

First, consider the second order Euler-Cauchy equation with a monomial right-hand side function,

\[
t^2 y'' + aty' + by = At^\alpha, \quad t > 0.
\]  

(1)

If we suppose that \( \alpha \in \mathbb{R} \) is not a root of the characteristic equation, then the above discussion indicates that we should try as our particular solution \( y_p = Ct^\alpha \). Plugging \( y_p \) into (1) gives

\[
(\alpha(\alpha - 1) + a\alpha + b)Ct^\alpha = At^\alpha.
\]
Since we have assumed that \( t > 0 \) and \( \alpha \) is not a root of the characteristic equation, we can solve directly for \( C \).

But, what if \( \alpha \) is, in fact, a root of the characteristic equation? As mentioned above, the Euler-Cauchy equation can be transformed into a constant-coefficient equation by means of the transformation \( t = e^z \). This means that our first guess for the particular solution would be \( y_p(z) = Ce^{\alpha z} \). But, since \( \alpha \) is a root of the characteristic equation, we need to multiply by \( z \) until \( y_p(z) \) is no longer a solution to the complementary equation. Multiplication by \( z \) in the guess for the particular solution for the transformed equation translates into multiplication by \( \ln(t) \) in the particular solution for (1), suggesting a particular solution of the form of a constant multiple of \( t^\alpha \) and a power of \( \ln(t) \). We can verify by direct substitution that this is the correct form of the solution.

These ideas are summarized in the following theorem.

**Theorem 1.** For the second order Euler-Cauchy problem,

\[
t^2y'' + aty' + by = At^\alpha, \quad t > 0,
\]

where \( \alpha \in \mathbb{R} \), a particular solution is of the form

(i) \( y_p(t) = Ct^\alpha \), provided that \( \alpha \) is not equal to any root of the characteristic equation, or

(ii) \( y_p(t) = Ct^\alpha (\ln(t))^i \), if \( \alpha \) is equal to a root of the characteristic equation, where \( i \) is the multiplicity of the root.

For the more complicated equation

\[
t^2y'' + aty' + by = At^\alpha (\ln(t))^n, \quad t > 0,
\]

where \( \alpha \in \mathbb{R} \) and \( n \) is a nonnegative integer, a similar analysis leads to the following theorem.

**Theorem 2.** For the second order Euler-Cauchy problem,

\[
t^2y'' + aty' + by = At^\alpha (\ln(t))^n, \quad t > 0,
\]

where \( \alpha \in \mathbb{R} \) and \( n \) is a nonnegative integer, a particular solution is of the form

\[
y_p(t) = (C_0 + C_1 \ln(t) + \cdots + C_n (\ln(t))^n) t^\alpha.
\]

In fact, the above method will lead to a solution using undetermined coefficients for the following types of functions, as well:

1. \( A \cos(k \ln t) \) or \( A \sin(k \ln t) \),
2. \( At^\alpha \cos(k \ln t) \) or \( At^\alpha \sin(k \ln t) \), and
3. \( At^\alpha (\ln(t))^n \cos(k \ln t) \) or \( At^\alpha (\ln(t))^n \sin(k \ln t) \).

You should, of course, verify this.

By the principle of superposition, the above results can be applied to Euler-Cauchy equations whose right-hand sides are sums of such functions, simply by applying the appropriate result to each term on the right-hand side. Here is an example.

**Example.** Find a general solution of \( t^2y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t, \quad t > 0 \).
• Complementary solution: Solve \( t^2 y'' - 4ty' + 4y = 0 \) to obtain \( y_c = c_1 t + c_2 t^4 \).

• Particular solution: Find a solution of \( t^2 y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t \).

The particular solution takes the form \( y_p = y_{p1} + y_{p2} \). Since the first function is \( 4t^2(\ln(t))^2 \), by Theorem 2 the first component of \( y_p \), \( y_{p1} \), is \( (A + B(\ln(t)) + C(\ln(t))^2) t^2 \). The particular solution corresponding to the second function, \( t \), is determined using Theorem 1. Since 1 is a simple root of the characteristic equation, the second component of \( y_p \), \( y_{p2} \), is \( Dt \ln(t) \). So, \( y_p = (A + B(\ln(t)) + C(\ln(t))^2) t^2 + Dt \ln(t) \). Plug \( y_p \) into the differential equation, collect terms, and equate coefficients to obtain \( A = -3, B = 2, C = -2, \) and \( D = \frac{1}{3} \), so

\[
y_p = (-3 + 2 \ln(t) - 2(\ln(t))^2) t^2 + \frac{1}{3} t \ln(t).
\]

General solution: \( y = y_c + y_p \), so

\[
y(t) = c_1 t + c_2 t^4 + (-3 + 2 \ln(t) - 2(\ln(t))^2) t^2 + \frac{1}{3} t \ln(t).
\]

It is straightforward to generalize the approach described in this paper to higher order Euler-Cauchy equations.

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Summary. The Euler-Cauchy equation is often the first higher order differential equation with variable coefficients introduced in an undergraduate differential equations course. Putting a non-homogeneous Euler-Cauchy equation on an exam in such a course, I was surprised when some of my students decided to apply the method of undetermined coefficients, supposedly guaranteed to work only for constant-coefficient equations, and obtained the correct answer! It turns out that a particular solution to this equation has a form similar to that of standard undetermined coefficients, if the right-hand side function is of a certain type. Thus the Euler-Cauchy equations can be solved without using variation of parameters or a substitution transforming the equation to a constant-coefficient equation.

Suspension Bridge Profiles

C.W. Groetsch (charles.groetsch@citadel.edu), The Citadel, Charleston SC 29409-6420 doi:10.4169/074683410X488737

The shape of a uniform flexible cable hanging by its own weight—the catenary—is an historically significant topic in an elementary differential equations course, often deemed too specialized for a first calculus course. However, the conceptually simpler case of a heavy uniform horizontal roadbed or deck suspended from a cable of insignificant mass—leading to a parabolic profile—is more tractable and can profitably be presented in the calculus classroom (e.g., [3]). Here we treat the slightly more general problem of a non-uniform deck, characterize the shape of the suspension cable, reveal the catenary to be a special case, and illustrate some other interesting properties of this model.