NOTES

Derangements and Bell Numbers

ROBERT J. CLARKE
MARTA SVED
University of Adelaide
Adelaide, South Australia

1. Introduction We will establish an interesting connection between derangements and Bell numbers. As preliminaries we summarize some well-known properties of these combinatorial numbers.

A permutation of a set of elements ranking from 1 to n is called *derangement* if none of the elements is left at its original place. A well-known example of this is the case of the absent-minded secretary who has to place n letters into n addressed envelopes and puts *each* letter into a wrong envelope. The problem of enumerating the derangements has a history of some 200 years. It was first posed by Pierre Rémond de Montmort (1678–1719) and was called "le problème de rencontres". The number of derangements of a set of n elements is denoted by $D(n)$. Thus $D(0) = 1$, $D(1) = 0$, $D(2) = 1$, $D(3) = 2$, $D(4) = 9$.

Bell numbers, denoted by $B(n)$, count the number of partitions of a set of n distinguishable objects into unordered non-empty subsets. Thus $B(1) = 1$, $B(2) = 2$, $B(3) = 5$, $B(4) = 15$ and by definition $B(0) = 1$. The Bell numbers are closely related to the well-known Stirling numbers of the second kind, $S(n, k)$. In fact, as $S(n, k)$ equals the number of partitions of a set of n distinguishable objects into k unordered nonempty subsets, we have

$$B(n) = \sum_{k=1}^{n} S(n,k), \quad n \geq 1.$$

Though these numbers are named after Eric Temple Bell (1883–1960), the problem of classifying and enumerating the partitions on n objects occupied mathematicians through much earlier periods. Euler made important contributions to the subject. We refer the reader to Biggs [1] and Stein [5] for further historical background, and to Stanley [4] for a modern treatment of Bell numbers, Stirling numbers, and derangements.

It is not difficult to find a closed formula for the number of derangements $D(n)$. It is obtained by the use of the inclusion-exclusion principle. Denote by $R = \{P_1, P_2, \ldots, P_r\}$ a set of properties that each of the elements of some set N may or may not possess. Denote by N_i the number of elements in N having property P_i, N_{ij} the number having properties P_i and P_j, and generally by $N_{ij\ldots k}$ the number of elements having all the properties $P_i, P_j, \ldots, P_k$, (and possibly other properties belonging to R).

Denote by N_0 the number of elements of N that have none of the properties listed in R. Then

$$N_0 = |N| - \sum_i N_i + \sum_{ij} N_{ij} + \cdots + (-1)^r \sum N_{12\cdots r} \tag{1}$$

The set to be considered now is the set of all the permutations of $\{1, 2, \ldots, n\}$. Then $|N| = n!$ Let property P_i be: The element i occupies place i, $(i \in \{1 \cdots n\})$. Then

$$\sum N_i = \binom{n}{1}(n-1)!, \qquad \sum N_{ij} = \binom{n}{2}(n-2)!$$

and generally

$$\sum N_{i_1 i_2 \cdots i_k} = \binom{n}{k}(n-k)!.$$

Thus by (1) we have

$$D(n) = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \cdots + (-1)^n.$$

This result can be written in the form

$$D(n) = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!}\right). \tag{2}$$

From here some interesting well-known properties of $D(n)$ follow immediately.

(i) $\dfrac{D(n)}{n!}$ converges quickly to e^{-1},

(ii) $D(n) = nD(n-1) + (-1)^n$, and

(iii) $D(n) = (n-1)(D(n-1) + D(n-2))$.

Although a nice combinatorial proof of (iii) can be given quite easily, (ii) is a different matter. We refer the reader to Remmel [2] for a rather difficult combinatorial proof of (ii). The two recursion formulae (ii) and (iii) allow fast evaluation of $D(n)$.

Recognizing that $\binom{n}{k}D(k)$ counts all permutations in which exactly k elements of the set $\{1, 2, \ldots, n\}$ are displaced, we obtain the relation

$$\sum_{k=0}^{n} \binom{n}{k} D(k) = n!. \tag{3}$$

Bell numbers are more elusive. They can be evaluated successively using an identity with a left-hand side similar to that of (3). The relation is

$$\sum_{k=0}^{n} \binom{n}{k} B(k) = B(n+1). \tag{4}$$

Here we note that $\binom{n}{k}B(n-k)$ counts all the partitions of the set $\{a_1, a_2, \ldots, a_{n+1}\}$ where the last element a_{n+1} appears in some subset containing exactly $k+1$ elements. Such a partition is constructed by finding k elements out of the set $\{a_1, a_2, \ldots, a_n\}$ sharing the subset with a_{n+1} and partitioning in some way the remaining set, hence the above result. Thus

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(n-k) = \sum_{k=0}^{n} \binom{n}{n-k} B(k) = \sum_{k=0}^{n} \binom{n}{k} B(k),$$

as stated in (4).

2. An identity connecting the numbers $D(n)$ and $B(n)$ The following problem has been posed by Terry Tao [6].

Prove the identity

$$\sum_{m=0}^{n} \sum_{i=0}^{m} \frac{(-1)^i m^2}{i!(n-m)!} = n^2 - 2n + 2. \tag{5}$$

Using equation (2), this identity can easily be arranged to

$$\frac{1}{n!} \sum_{m=0}^{n} m^2 \binom{n}{m} D(m) = n^2 - 2n + 2. \tag{6}$$

However, we note here that the identity is not valid for $n < 2$.

We may look at (6) in probabilistic terms. Over the sample space of all the permutations of n elements define the random variable X as the number of displaced elements. Then the probability

$$P(X = m) = \binom{n}{m} \frac{D(m)}{n!}.$$

Thus equation (3) may be interpreted as

$$\sum_{m=0}^{n} \binom{n}{m} \frac{D(m)}{n!} = \sum_{m=0}^{n} P(X = m) = 1.$$

Identity (6) states that the second moment of X about the origin is given by the quadratic $n^2 - 2n + 2$.

It is also easy to show that the first moment of X, that is its expected value, is

$$\sum_{m=0}^{n} m \binom{n}{m} \frac{D(m)}{n!} = n - 1.$$

Writing $P_0(n) = 1$, $P_1(n) = n - 1$ and $P_2(n) = n^2 - 2n + 2$, (assuming the validity of the last one), it can be expected that $P_s(n)$, the s-th moment of X about the origin, is a polynomial in n of degree s.

In fact, assuming that

$$\sum_{m=0}^{n} m^{s-1} \binom{n}{m} D(m) = n! P_{s-1}(n) \quad \text{for } n \geq s - 1, \tag{7}$$

where $P_{s-1}(n)$ is a polynomial of degree $s - 1$, (established for degrees 0 and 1), we may write

$$\sum_{m=0}^{n} m^s \binom{n}{m} D(m) = \sum_{m=0}^{n} (n - (n - m)) m^{s-1} \binom{n}{m} D(m).$$

Using

$$(n - m)\binom{n}{m} = n\binom{n-1}{m},$$

we get

$$\sum_{m=0}^{n} m^s \binom{n}{m} D(m) = n \cdot n! P_{s-1}(n) - n \sum_{m=0}^{n-1} m^{s-1} \binom{n-1}{m} D(m)$$

$$= n!(nP_{s-1}(n) - P_{s-1}(n-1)) = n!P_s(n),$$

where

$$P_s(n) = nP_{s-1}(n) - P_{s-1}(n-1). \tag{8}$$

The recursion (8) shows that the sth moment of X is indeed a polynomial of degree s in n.

Our aim in this note is to give a representation of the polynomials $P_s(n)$ in terms of Bell numbers:

$$P_s(n) = \sum_{j=0}^{s} (-1)^j \binom{s}{j} n^{s-j} B(j) \quad \text{for } n \geq s,$$

or equivalently, using (7),

$$\sum_{m=0}^{n} m^s \binom{n}{m} D(m) = n! \sum_{r=0}^{s} (-1)^j \binom{s}{j} n^{s-j} B(j) \quad \text{whenever } n \geq s. \tag{9}$$

We will give an *elementary* proof of relation (9). Before doing so, let us note that one can prove (9) algebraically, using generating functions. Namely, one can use (8) and, after a reasonable amount of work, construct generating functions for $P_s(n)$:

$$\sum_{s=0}^{\infty} P_s(n) x^s / s! = e^{nx} \exp(e^{-x} - 1).$$

Comparing these functions with the well-known generating function for the Bell numbers,

$$\sum_{n=0}^{\infty} B(n) x^n / n! = \exp(e^x - 1),$$

one obtains (9).

We note that C. C. Rousseau [3] has also generalised Terry Tao's identity, obtaining an equivalent result to (9) by a somewhat different algebraic route.

3. Combinatorial interpretation of identity (9) Let N be the set of permutations of $\{1, 2, \ldots, n\}$ as before. In addition we define a set of symbols

$$S = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}, \quad \text{where } s \leq n$$

such that *each* symbol of S is associated with *exactly one* element a_i in some given permutation $a_1 a_2 \ldots a_n$ in N, subject to the following *restriction*.

A symbol of S may only be associated with an element a_i if a_i is a displaced element, that is $a_i \neq i$. (Note that more than one symbol of S may be associated with the same element a_i.)

We consider first the left-hand side of (9). Suppose that there are exactly m displaced elements in a certain permutation belonging to N. There are m^s ways in which each of the symbols can be associated to one of the m elements. The factor $\binom{n}{m} D(m)$ gives the number of permutations with m displaced elements. *Hence the sum from $m = 0$ to n counts all possible associations of the symbols S with the permutations belonging to N under the stated restriction.*

Next we use the inclusion-exclusion principle to show that the right-hand side of (9) gives the same count.

Disregarding first the restriction, there are $A = n^s n!$ ways in which the symbols belonging to S can be associated with the permutations belonging to N. However, by the convention adopted, an *error* occurs whenever at least one of the symbols is attached to an element a_i where $a_i = i$. Denote by A_j the sum of the numbers of associations, in which j *specified errors occur*, and accordingly by A_0 the number of "legitimate" associations with no errors. Then by (1)

$$A_0 = A - A_1 + A_2 + \cdots + (-1)^j A_j + \cdots + (-1)^s A_s. \tag{10}$$

To determine A_j assume first that (at least) j symbols are attached to l elements that are not displaced, where

$$1 \le l \le j \quad (\text{we assume here that } l \le n).$$

There are $\binom{n}{l}$ ways of choosing these fixed elements. The number of ways in which j specified symbols can be divided between l ordered places is

$$l!S(j,l)$$

where $S(j,l)$ is the Stirling number of the second kind. The remaining $s-j$ symbols can be allocated freely to any place in the permutation considered, hence in n^{s-j} ways. Since there are $(n-l)!$ permutations where at least l elements are fixed, the number of illegal associations in which *at least* each of the specified j symbols is consigned to exactly l fixed elements is

$$n^{s-j} l! S(j,l) \binom{n}{l} (n-l)! = n! S(j,l) n^{s-j}. \tag{11}$$

Summing for $l = 1$ to s and for all possible choices of j symbols out of the set S we obtain

$$A_j = \binom{s}{j} \sum_{l=1}^{s} n! n^{s-j} S(j,l) = n! \binom{s}{j} n^{s-j} B(j),$$

since

$$\sum_{l=1}^{s} S(j,l) = B(j).$$

(We note here that if $s > n$, then at least one of the l values in the range $\{1, 2, \ldots, s\}$ is greater than n, and so we cannot arrive at the left-hand side of (11) and our reasoning breaks down.)

Substituting into (1) we obtain the right hand side of (9). This completes the proof. It is easy to check that the identity is valid only for $n \ge s$.

REFERENCES

1. N. L. Biggs, The roots of combinatorics, *Historia Math.* 6 (1979), 109–136.
2. J. B. Remmel, A note on a recursion for the number of derangements, *European J. Combinatorics* 4 (1983), 371–374.
3. C. C. Rousseau, Binomial identity 31, *James Cook Mathematical Notes* 5 (1990), 5252–5253.
4. Richard P. Stanley, *Enumerative Combinatorics Volume I*, Wadsworth & Brooks/Coke, Monterey, CA, 1986.
5. Paul R. Stein, A brief history of enumeration, *Advances in Mathematics Supplementary Studies* 10 (1986), 169–206.
6. Terry Tao, Binomial identity 31, *James Cook Mathematical Notes* 5 (1990), 5227.