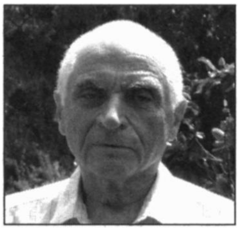
Do Dogs Know Calculus of Variations?

Leonid A. Dickey



Leonid Dickey (ldickey@math.ou.edu) was born in Kharkov (now in Ukraine) in 1926. He received the degrees of Candidate and Doctor of Physical and Mathematical Sciences (the former approximately equivalent to a Ph.D.) from Moscow State University, where he worked for many years. He came to the United States in 1988 and held visiting positions at MIT, Brandeis, and the University of North Carolina before settling at the University of Oklahoma (Norman, OK 73019). In addition to mathematics and his family, his interests include classical music and hiking.

The article titled "Do dogs know calculus?" by T. J. Pennings published in this magazine **[3]** discusses a problem with far-reaching generalizations. Pennings compared his dog's behavior with the mathematical solution to an optimization problem. Recall the scenario: the dog is at the water's edge on a beach, and a tennis ball is thrown into the water. What should be the strategy of the dog if it is eager to reach the ball in the shortest time?

In this note, we begin by considering a slight generalization of the problem, assuming that the dog stands not at the edge of the water but some distance from the shore, as indicated in Figure 1. If the running speed of the dog on the beach is r and its swimming speed is s, then, with the notation in Figure 1 (and $D'B' = c$), we find that the time it takes the dog to reach the ball is

$$T = \frac{\sqrt{a^2 + x^2}}{r} + \frac{\sqrt{b^2 + (c - x)^2}}{s}.$$

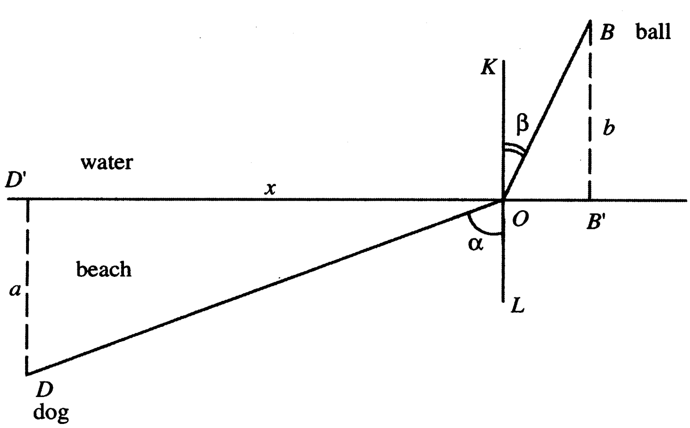


Figure 1. The fastest way

In order to find the minimum time, we set the derivative (taken with respect to x) equal to 0:

$$\frac{x}{r\sqrt{a^2+x^2}} - \frac{c-x}{s\sqrt{(c-x)^2+b^2}} = 0.$$

Since

$$\sin\alpha = \frac{x}{\sqrt{a^2+x^2}} \quad \text{and} \quad \sin\beta = \frac{c-x}{\sqrt{(c-x)^2+b^2}},$$

the optimal course satisfies the equation

$$\frac{\sin\alpha}{\sin\beta} = \frac{r}{s}. \tag{1}$$

Let us leave the dog now and turn our attention to optics. Fermat's principle states that light going from one place to another follows the quickest route. If Figure 1 is interpreted as representing two media (instead of sand and water) in which light travels at speeds r and s, then equation (1) is nothing more than Snell's law.

However, this is not the whole story. The dog problem has a surprising application to the classical brachistochrone problem in mechanics posed by Johann Bernoulli at the end of the 17th century and solved by him with the help of his brother Jacob. Suppose there are two points A and B in a vertical plane, A lies higher than B, but not directly above. Under only the influence of gravity, a particle rolls along a curve connecting A and B in this plane. (No friction is taken into account.) The problem is to find the shape of the curve along which the particle will travel in the shortest time. (The initial velocity is assumed to be zero.)

The brachistochrone problem was one of the first, if not the very first, problem of what is called the calculus of variations. This is a branch of mathematics that studies maxima and minima of functions that do not depend only on variables taking numerical values, as in the ordinary calculus. The functions in the calculus of variations (also called "functionals") depend on curves. For example, a problem might be to find the curve on a surface connecting two given points that has minimum length. Here, the functional is the length of the curve. In the brachistochrone problem, the functional is the time of descent of the particle along the curve.

When the height of the particle decreases by y, it loses the potential energy mgy, which is transformed into the kinetic energy $mv^2/2$. Thus, $v = \sqrt{2gy}$. Let us approximate the curve by a broken line (Figure 2). The space will be divided into strips $[y_0, y_1]$, $[y_1, y_2]$, ... The velocity in the ith strip is approximately constant, $v_i = \sqrt{2gy_i}$. If the trajectory is optimal, then it must be optimal for all the strips, in par-

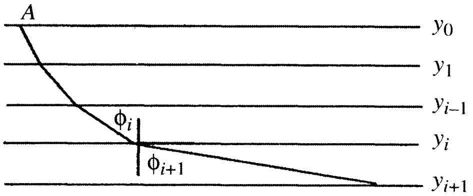


Figure 2. Approximation of the brachistochrone

ticular for consecutive ones. Therefore, Snell's law can be applied:

$$\frac{\sin\phi_{i+1}}{\sin\phi_i} = \frac{v_{i+1}}{v_i} = \frac{\sqrt{y_{i+1}}}{\sqrt{y_i}}$$

(the angles ϕ_i are shown in Figure 2). The strips have constant width h. Then

$$\frac{\sin^2\phi_{i+1}}{\sin^2\phi_i} = 1 + \frac{h}{y_i}.$$

Taking logarithms, we get

$$2(\log\sin\phi_{i+1} - \log\sin\phi_i) = \log\left(1 + \frac{h}{y_i}\right) \approx \frac{h}{y_i},$$

which, after we divide by h becomes

$$2\frac{\log\sin\phi_{i+1} - \log\sin\phi_i}{h} \approx \frac{1}{y_i}.$$

Now we pass to the limit, letting the width h of the strips go to 0 (and thus the number of strips go to infinity). The broken line then becomes a smooth curve $x = x(y)$. Let $i \to \infty$ so that $ih \to y$. Then $\phi_i \to \phi$, where ϕ is the angle between the tangent line to the curve and the y-axis. We obtain the differential equation

$$2\frac{d}{dy}\log\sin\phi = \frac{1}{y}.$$

This equation can be integrated, giving

$$2\log\sin\phi = \log y + C_1,$$

whence

$$\sin^2\phi = Cy. \qquad (2)$$

We now show that (2) describes a cycloid, the curve traced out by a point on the circumference of a circle as the circle rolls along a straight line (see Figure 3). Two positions of the circle (of radius R) are shown, A being the starting point of the trajectory. We choose as a parameter the angle of rotation s of the circle. The point on the circumference of the circle that coincided with A at the beginning is A' in a new position of the circle. The length of AC equals that of arc CA'. The point C on the circle is immobile at the moment shown on the picture; it is the instantaneous center of rotation. Therefore, the tangent line to the trajectory must be perpendicular to CA', and the angle ϕ is that shown in the figure. One can see that ϕ is equal to $\angle COK$ because the two angles have parallel sides, and so $\phi = s/2$. Finally,

$$y = |CL| = R + R\cos(\pi - s) = R(1 - \cos s) = R(1 - \cos 2\phi) = 2R\sin^2\phi,$$

which is nothing but equation (2) with $C = (2R)^{-1}$. The constant C (or, equivalently, the radius R) can be found from the condition that the trajectory passes through the second given point, B. It can be proven that there is one and only one cycloid starting

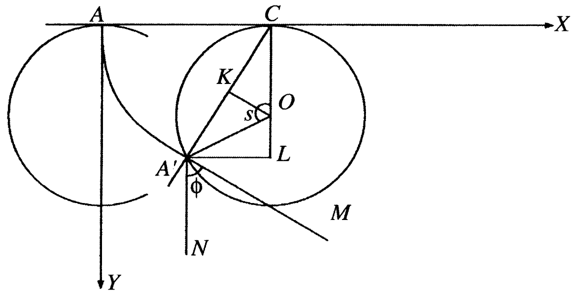


Figure 3. Derivation of the property (2) of the cycloid

at A and passing through any point B lower than A (see [**1**, p. 55]). Thus, the curve of fastest descent from the point A to the point B is a cycloid.

This was essentially the way Bernoulli solved the problem. It was an ad hoc method that worked only for this particular problem. Later, Leonhard Euler suggested a general method for writing differential equations for problems in the calculus of variations. His method became standard, and its application to the brachistochrone problem can be found in [**2**].

As to the dog, it is interesting, would it choose the optimal path if the properties of the soil (density or humidity and, correspondingly, the running speed) change gradually?

Added in proof: We note that equation (2) can be derived more simply using Snell's law $\frac{\sin\varphi_{i+1}}{\sin\varphi_i} = \frac{\sqrt{y_{i+1}}}{\sqrt{y_i}}$: Simply rearrange terms to get $\frac{\sin\varphi_{i+1}}{\sqrt{y_{i+1}}} = \frac{\sin\varphi}{\sqrt{y_i}}$. This means that $\frac{\sin\varphi_i}{\sqrt{y_i}}$ is constant, so (2) follows by taking the limit as $h \to 0$.

References

1. Gilbert A. Bliss, *Calculus of Variations*, MAA, 1944.
2. Nil P. Johnson, The Brachistochrone Problem, *College Math. J.* **35** (2004) 192–197.
3. Timothy J. Pennings, Do Dogs Know Calculus?, *College Math. J.* **34** (2003) 178–182.

Stephen Wirkus (swirkus@csupomona.edu), Jennifer Switkes, and Randall Swift, of California State Polytechnic State University Pomona, wrote to point out that the paper "Highway Relativity" (*College Math. J.* 35 (2004) 246–250) is incorrect in saying "when you travel slower than the average speed of traffic, your perception of that average will be exaggerated." This claim, which has been put forth by others, is true for symmetric distributions. However, they show in their paper "Perceived Highway Speed" (*Math. Scientist* 28 (2003) 28–36) that this claim is not true in general for non-symmetric distributions. That is, one can be moving slower than the mean and yet perceive the average speed as even slower, and vice versa. Further examples of these phenomena can be found in their forthcoming letter to the editor "On Highway Relativity" (*Math. Scientist* 31 (2006)).