Further exploration What derivative sign patterns are there for other domains in $\mathbb{R}$, such as $(0, \infty)$? What if we extend the definition of derivative sign patterns to functions of more than one variable and include partial derivatives? These are just a few of the open questions worth exploration.

Summary. Analysis of the patterns of signs of infinitely differentiable real functions shows that only four patterns are possible if the function is required to exhibit the pattern at all points in its domain *and* that domain is the set of all real numbers. On the other hand *all* patterns are possible if the domain is a bounded open interval.

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Limit Interchange and L'Hôpital's Rule

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Students eventually learn the importance and unifying power of interchanging limits with other operations, like summation and integration, through advanced results like Lebesgue's Dominated Convergence Theorem and Fubini's Theorem. Even in an introductory analysis course, limit interchange shows up in connection with uniform convergence. Can the concept of limit interchange be introduced still earlier—in a first-year calculus course? L'Hôpital's rule provides an opportunity.

A geometric sum This example, though simple, is characteristic. The goal is to add a finite number of 1's:

$$\begin{aligned} 1 + 1 + \cdots + 1 \ (n \text{ terms}) &= 1 + \lim_{r\to 1} r + \lim_{r\to 1} r^2 + \cdots + \lim_{r\to 1} r^{n-1} \ (n \text{ terms}) \\ &= \lim_{r\to 1} \left(1 + r + \cdots + r^{n-1}\right) = \lim_{r\to 1} \left(\frac{r^n - 1}{r - 1}\right) \qquad (1) \\ &= \lim_{r\to 1} \left(\frac{nr^{n-1}}{1}\right) = n \end{aligned}$$

Passage from the first line to the second is by interchanging the sum with a limit; passage to the last line is by l'Hôpital's Rule. The final result is ridiculously obvious—comical, but also crucial, because it proves the validity of the limit interchange.

Limits under the integral sign Integrals containing a parameter are a natural domain of application for l'Hôpital's Rule. Here is one example:

$$\int \frac{dt}{t^2 + a^2} = \begin{cases} \dfrac{1}{a} \arctan\left(\dfrac{t}{a}\right) + C & a > 0 \\ -\dfrac{1}{t} + C & a = 0, \end{cases}$$

where t must be non-negative. Is it permissible to pass from $a > 0$ to $a = 0$ by a limit under the integral sign? The convergence here (as a approaches 0) is uniform,

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monotonic and dominated, but there is no need for the corresponding limit interchange theorems. L'Hôpital's Rule is sufficient. Converting to a definite integral (to control the constant of integration):

$$\begin{aligned}
\frac{1}{x} &= \int_x^\infty \frac{1}{t^2}\,dt = \int_x^\infty \lim_{a\to 0^+}\left(\frac{1}{t^2+a^2}\right)dt \\
&= \lim_{a\to 0^+}\int_x^\infty \left(\frac{1}{t^2+a^2}\right)dt = \lim_{a\to 0^+}\left[\frac{\frac{\pi}{2}-\arctan\left(\frac{x}{a}\right)}{a}\right] \qquad (2)\\
&= \lim_{a\to 0^+}\frac{xa^{-2}}{1+x^2a^{-2}} = \lim_{a\to 0^+}\frac{x}{a^2+x^2} = \frac{1}{x}.
\end{aligned}$$

As with (1), passage to the second line is by limit interchange, while the third line is an application of l'Hôpital's Rule. Again, final equality proves that limit interchange is valid.

Calculus students can be asked to justify taking a limit under the integral sign by imitating (2). Examples abound in tables of integrals. Here are two more, whose verification is left to the reader:

$$\int t^n\,dt = \begin{cases} \dfrac{t^{n+1}}{n+1}+C & n\neq -1 \\ \ln(t)+C & n=-1 \end{cases}$$

and

$$\int \cos(nt)\,dt = \begin{cases} \dfrac{1}{n}\sin(nt)+C & n\neq 0 \\ t+C & n=0, \end{cases}$$

where $t > 0$ in the first integral. This kind of problem not only reinforces technical skills (integration *and* l'Hôpital's rule) but gives a taste of the kind of general principle that underlies more advanced work in analysis.

A counter-example This is based on Rudin [**1**, p. 146]:

$$\begin{aligned}
0 &= \int_0^1 0\,dt = \int_0^1 \lim_{n\to\infty} nt\left(1-t^2\right)^n\,dt \\
&= \lim_{n\to\infty}\int_0^1 nt\left(1-t^2\right)\,dt = \lim_{n\to\infty}\frac{n}{2n+2} = \frac{1}{2}!
\end{aligned}$$

Thus limit interchange is *not* always justified. Negative examples are vital. They point to the need for a general theory and keep students on their toes.

Summary. Conventional application of these two calculus staples is stretched here, somewhat recreationally, but also to raise solid questions about the role of limit interchange in analysis—without, however, delving any deeper than first-year Calculus.

References

1. W. R. Rudin, *Principles of Mathematical Analysis* (Third edition), McGraw-Hill, New York, 1976.