EDITORS

Ricardo Alfaro
University of Michigan–Flint
Flint MI 48502
ralfaro@umflint.edu

Lixing Han
University of Michigan–Flint
Flint MI 48502
lxhan@umflint.edu

Kenneth Schilling
University of Michigan–Flint
Flint MI 48502
ksch@umflint.edu

Classroom Capsules are short (1–3 page) notes that contain new mathematical insights on a topic from undergraduate mathematics, preferably something that can be directly introduced into a college classroom as an effective teaching strategy or tool. Classroom Capsules should be prepared according to the guidelines on the inside front cover and sent to any of the above editors.

The Product and Quotient Rules Revisited

Roger Eggleton (roger@ilstu.edu) and Vladimir Kustov (vmkoust@ilstu.edu) Mathematics Department, Illinois State University, Normal IL 61790

Let's teach mathematical elegance! Elegance should not be neglected in the classroom, even when a more prosaic formulation might have some practical advantage. We illustrate this principle by considering the product and quotient rules of basic calculus:

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}, \tag{P}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}. \tag{Q}$$

These are typical of the formulations in introductory calculus texts (such as [**1**], [**2**]), but better students can also be shown the alternative versions:

$$(fg)' = fg\left(\frac{f'}{f} + \frac{g'}{g}\right), \tag{P$'$}$$

$$\left(\frac{f}{g}\right)' = \frac{f}{g}\left(\frac{f'}{f} - \frac{g'}{g}\right). \tag{Q$'$}$$

The parallelism between (P′) and (Q′) appeals strongly to our sense of mathematical elegance, a feature which (P) and (Q) distinctly lack. Such parallelism is also a real aid to memorization.

It is easy to establish (P′) and (Q′) by logarithmic differentiation. They can also be obtained by forcing (P) and (Q) into a more uniform shape. But in either case, note that f and g must be nonzero for (P′) and (Q′) to hold.

If one function is replaced by a product of functions in (P′) or (Q′), we get

$$(fgh)' = fgh\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right),$$

$$\left(\frac{f}{gh}\right)' = \frac{f}{gh}\left(\frac{f'}{f} - \frac{g'}{g} - \frac{h'}{h}\right),$$

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and so on. With (P′) and (Q′) it is also straightforward to differentiate a product or quotient which already contains one or more derivatives, such as

$$\left(\frac{f'}{f}\right)' = \frac{f'}{f}\left(\frac{f''}{f'} - \frac{f'}{f}\right).$$

Readers may enjoy using this line of reasoning to show, for example, that

$$\left(\frac{f}{f'}\right)'\left(\frac{f'}{f}\right)' + \left(\frac{f''}{f'} - \frac{f'}{f}\right)^2 = 0.$$

Stationary points If (P′) and (Q′) are presented in the classroom together with (P) and (Q), how should stationary points of products and quotients be determined? They conveniently follow from one of the equivalences

$$(fg)' = 0 \iff f'g + fg' = 0,$$
$$\left(\frac{f}{g}\right)' = 0 \iff f'g - fg' = 0.$$

The parallelism between these two expressions again strongly appeals to our sense of mathematical elegance and serves as a memory aid. This time it is (P) and (Q) that most readily yield these expressions. They could also be deduced from (P′) and (Q′), though this route would restrict the discussion to nonzero f and g, an issue of marginal importance theoretically, but of potential significance in any concrete computation. (Note that f/g is not defined where g is zero, so this constraint is implicit in any discussion of the quotient.)

For example, (Q′) quickly gives us

$$\left(\frac{x^2}{\sin x}\right)' = \frac{x^2}{\sin x}\left(\frac{2}{x} - \cot x\right),$$

but the stationary points of $x^2/\sin x$ are best obtained from

$$2x\sin x - x^2\cos x = 0 \quad \text{with} \quad \sin x \neq 0.$$

Since $x^2/\sin x$ is not defined at $x = 0$, its stationary points are the nonzero roots of the transcendental equation $2\tan x = x$.

Some applications In contexts where relative rates of change are of interest, (P′) and (Q′) have natural interpretations, since they are effectively about such rates. Let us take some examples from economics.

If f is unit price and g is the quantity demanded, then fg is revenue. Hence, revenue is maximized when

$$\frac{(fg)'}{fg} = 0, \quad \text{so} \quad \frac{f'}{f} + \frac{g'}{g} = 0.$$

Economists call the quotient $\frac{g'}{g}/\frac{f'}{f}$ the price elasticity of demand, and use the preceding calculation to deduce the total revenue test [**3**]: revenue is maximized when price elasticity of demand is -1.

Again, if f is gross domestic product (GDP) and g is population, then f/g is gross domestic product per capita. If GDP increases by 2% and population increases by 3%, then gross domestic product per capita *decreases* by approximately 1%, since

$$\frac{(f/g)'}{f/g} = \frac{f'}{f} - \frac{g'}{g}$$

implies the incremental relationship

$$\frac{\Delta(f/g)}{f/g} \approx \frac{\Delta f}{f} - \frac{\Delta g}{g} = 2\% - 3\% = -1\%.$$

Similarly, if f is total cost of producing g units of a product, then f/g is unit cost (or average cost). For instance, if total cost is seen to have remained constant while the number of units produced increased by a small percentage, then unit cost must have decreased by approximately the same percentage.

Second derivatives Higher derivative formulas also have elegant forms. For the second derivative of a product, we use (Q′) to differentiate (P′) in the form

$$\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g},$$

obtaining the second derivative product rule

$$\frac{(fg)''}{fg} = \frac{f''}{f} + \frac{g''}{g} + 2\frac{f'}{f} \cdot \frac{g'}{g}, \tag{P$''$}$$

corresponding naturally to the second derivative case of Leibniz's rule for higher derivatives of a product. Similarly, differentiating (Q′) in the form

$$\frac{(f/g)'}{f/g} = \frac{f'}{f} - \frac{g'}{g}$$

yields the second derivative quotient rule, in explicit or in iterative form:

$$\begin{aligned} \frac{(f/g)''}{f/g} &= \frac{f''}{f} - \frac{g''}{g} - 2\left[\frac{f'}{f} - \frac{g'}{g}\right]\frac{g'}{g} \\ &= \frac{f''}{f} - \frac{g''}{g} - 2\frac{(f/g)'}{f/g} \cdot \frac{g'}{g}. \end{aligned} \tag{Q$''$}$$

Comparison of the final terms in (P″) and (Q″) shows the greater complexity of the quotient formula. This helps to explain why Leibniz's rule is not routinely accompanied by an equivalent quotient rule. Readers may find it an interesting challenge to discover the elegant third derivative quotient rule for themselves. For enthusiasts, the corresponding higher derivative quotient rules await.

Summary. Mathematical elegance is illustrated by strikingly parallel versions of the product and quotient rules of basic calculus, with some applications. Corresponding rules for second derivatives are given: the product rule is familiar, but the quotient rule is less so.

References

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A Generalization of the Parabolic Chord Property

John Mason (j.h.mason@open.ac.uk) Oxford, England

It is well known that the tangents at either end of a chord of a parabola meet in a point aligned vertically with the midpoint of the chord. In other words, the point of intersection of the tangents at the two ends of a chord on a parabola and the midpoint of that chord lie on a line parallel to the axis of the parabola. In this JOURNAL, Stenlund [**3**] showed that this midpoint property characterizes quadratic polynomials; then Krasopoulos [**1**] extended the property to R^n; and Xu [**4**] showed that the property characterizes quadratics in R^n. The property has a natural generalization.

Theorem. *Given two distinct points on a polynomial of degree d, the Taylor polynomials of degree $d-1$ at those points meet in $d-1$ points whose mean is vertically aligned with the midpoint of the chord joining the two points.*

Proof. Without loss of generality, translate the origin to one end of the chord. Let the degree d polynomial be

$$p(x) = \sum_{k=1}^{d} c_k x^k,$$

where $c_d \neq 0$, then the Taylor polynomial of degree $d-1$ at the origin is simply

$$T_0(x) = \sum_{k=1}^{d-1} c_k x^k.$$

Let $(t,\, p(t))$ be another point on the graph of the polynomial. Then the Taylor polynomial at t is

$$T_1(x) = \sum_{k=0}^{d-1} \frac{p^{(k)}(t)}{k!}(x-t)^k.$$

The two "tangential functions" T_0 and T_1 are both of degree $d-1$, and so, in general, they meet in $d-1$ points (some of which may be complex) whose first coordinates are the roots of $D(x) = T_1(x) - T_2(x)$. The sum of the roots of D is, apart from sign, the coefficient of x^{d-2} divided by the coefficient of x^{d-1}.

One way to find the terms of T of degree k, avoiding derivatives, is to collect $p(x+t)$ in powers of x, delete the powers of x greater than k, and then substitute $x-t$ for

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