Schur Positivity of Differences of Products of Schur Functions

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Abstract

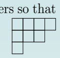
The Schur functions s_λ are a basis for the ring of symmetric functions indexed by partitions of nonnegative integers. A symmetric function is called Schur positive if when expressed as a linear combination of Schur functions, each coefficient is nonnegative. We investigate expressions of the form

$$s_\lambda s_{\lambda^c} - s_\mu s_{\mu^c}$$

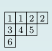
where λ partitions n and μ partitions $n-1$ and the complements λ^c, μ^c are taken over a sufficiently large $m \times m$ square. We first give a necessary condition for when this expression is Schur positive. We then provide symmetries and equivalences involved in the Schur positivity of our expression. Lastly, we incorporate the Littlewood Richardson rule to conjecture a full characterization of when $s_\lambda s_{\lambda^c} - s_\mu s_{\mu^c}$ is Schur positive.

Preliminaries

1 Partitions and Young Diagrams

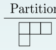
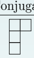
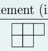
A *partition* $\lambda = (\lambda_1, \ldots, \lambda_r)$ of a positive integer n is a weakly decreasing list of non-negative integers so that the sum of the elements is n. We use Young diagrams to visualize partitions: $(4,3,1) \vdash 8 \quad \rightarrow$ 

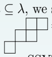
A Young tableau is a Young diagram with positive integers entered in the boxes. A *semistandard Young tableau* (SSYT) is a special type of tableau where the numbers are arranged weakly increasing along rows and strictly increasing along columns. The *type* of a SSYT, T, is the sequence $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_t)$ where α_i is the number of i's in T.

 has type (2,2,1,1,1,1) and shape (4,3,1)

Operations on λ:

- *conjugate* (λ'): partition obtained by transposing λ like a matrix
- *complement* (λ^c): partition obtained by taking all the remaining boxes in a rectangle that fits around λ.
- *containment*: $\mu \subseteq \lambda$ if Young diagram of μ fits inside Young diagram of λ.

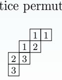
Partition	Conjugate	Complement (in 3×3)
		

If $\mu \subseteq \lambda$, we say λ/μ is the *skew shape* formed by removing μ from top left corner of λ.
Ex:  $= (4,3,1)/(2,1)$.

Given a SSYT T, define

- *reverse reading word* rw(T) : word $a_1 a_2, \ldots a_p$ obtained by reading the integers in T from right to left and top to bottom, row by row
- *lattice permutation*: word with at least as many occurrences of i as $i+1$ in subword $a_1 \ldots a_l$ for all $i, l \geq 1$.

A *Littlewood-Richardson Tableau* (LRT) is a skew semistandard tableau in which the reverse reading word is a lattice permutation.

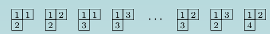
 is a LRT of shape $(4,3,2,1)/(2,1)$, type $(3,2,2)$, $rw(T) = 1121323$

2 Schur Functions

A symmetric function is invariant under any permutation that permutes only a finite number of variables, e.g. $f(x_1, x_2, \ldots) = x_1 + x_2 + \ldots$. A *Schur function* is a symmetric function, indexed by a partition λ, given by

$$s_\lambda = \sum_{\substack{T\ SSYT \\ of shape\ \lambda}} x^{type(T)} \quad , \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

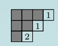
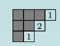
Ex:

$\lambda = (2,1):$  $\ldots$

$$\implies s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \ldots + 2x_1x_2x_3 + 2x_1x_2x_4 + \ldots$$

Schur Functions form basis for ring of symmetric functions. So, multiply Schur functions by expanding as linear combination of Schur functions : $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$, where $c^\nu_{\lambda\mu} = \#\{\text{LRT w/ shape } \nu/\mu \text{ and type } \lambda\}$.

$c^{(4,3,2)}_{(2,1),(3,2,1)} = 2$, since 2 LRT's of shape $(4,3,2)/(3,2,1)$ and type $(2,1)$

 and 

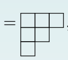
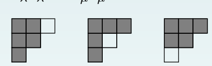
Our Problem

Let $\lambda \vdash n$ and $\mu \vdash n-1$. Let all complements be taken in an $m \times m$ box. When is $s_\lambda s^c_\lambda - s_\mu s^c_\mu$ Schur positive?

Results

1 A Necessary Condition for Schur Positivity: Containment

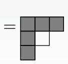
Proposition 1.1. *$s_\lambda s_{\lambda^c} - s_\mu s_{\mu^c}$ is Schur positive $\implies \mu \subseteq \lambda$*

- Q: If $\lambda =$ , what are the possibilities of $\mu \vdash 5$ such that $s_\lambda s_{\lambda^c} - s_\mu s_{\mu^c}$ could be Schur positive?
- A: 

If $\mu \subseteq \lambda$, leads us to

Definition 1.2. *(λ, μ) is a* paired k-hook *if $\mu \subseteq \lambda$ and*

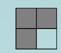
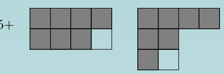
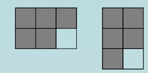
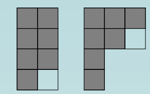
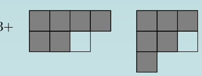
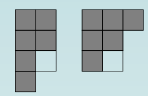
1. *$\mu_k = \lambda_k - 1$*
2. *$\mu_j = \lambda_j$ for all $j \neq k$*

- $(\lambda, \mu) =$  is a paired 2-hook.

Notation 1.3. *Write "(λ, μ) Schur positive" to mean $s_\lambda s_{\lambda^c} - s_\mu s_{\mu^c}$ is Schur positive.*

2 Symmetries and Equivalences

- Cases where $\mu \subseteq \lambda$, but $s_\lambda s_{\lambda^c} - s_\mu s_{\mu^c}$ is not Schur positive.

n	m	Non-SP	n	m	Non-SP
4	7+		8	15+	
6	11+				
7	13+				
					

Above table and other data illustrate following propositions

Proposition 2.1. (λ, μ) *is (not) Schur positive for all* $m \geq 2\max\{\lambda_1, \ell(\lambda)\} \iff (\lambda, \mu)$ *is (not) Schur positive for* $m = 2\max\{\lambda_1, \ell(\lambda)\}$

Proposition 2.2. (λ, μ) *Schur positive* $\iff (\lambda', \mu')$ *Schur positive*

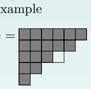
3 Sufficient Conditions

The goal is to characterize when any paired k-hook is Schur positive. We conjecture the following:

Conjecture 3.1. *Let (λ, μ) be a paired k-hook and let $m \geq 2\max\{\lambda_1, \ell(\lambda)\}$. Then,*

$$s_\lambda s_{\lambda^c} - s_\mu s_{\mu^c} \text{ is Schur positive} \iff \mu_{k-j} - \mu_k > \mu_k - \mu_{k+j} \quad \text{for all } 1 \leq j < k$$

We have proven the forward implication, but the backwards implication remains to be proven. As an example

- $(\lambda, \mu) =$  $k = 3, \quad 5 - 3 > 3 - 2$ and $6 - 3 > 3 - 1 \implies (\lambda, \mu)$ Schur positive

Future Work

- Complete the proof of (1).
- Reduce our bound of $m \geq 2\max\{\lambda_1, \ell(\lambda)\}$
- Explore the case when $\lambda \vdash n$ and $\mu \vdash p$ for $p \leq n$.

Acknowledgements

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