On the Measure of Solid Angles

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The measure of a solid angle in 3-space is defined to be the area E of the corresponding spherical triangle $T = ABC$ on the unit sphere, with the center at the vertex O of the angle (FIGURE 1). We denote the unit vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, respectively. The measure E can then be expressed in terms of dot products and the triple product $[\mathbf{a},\mathbf{b},\mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$:

$$\tan\frac{E}{2} = \frac{|[\mathbf{a},\mathbf{b},\mathbf{c}]|}{1+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}}. \tag{1}$$

One would think that such a nice formula should be old and well-known. But so far, I have not been able to find it in the literature. However, almost the same result was known to Euler [4, p. 215] and Lagrange [8, p. 340]. (The title above is a translation of the Latin title of [4].) We use the standard notations a, b, c for the sides of our spherical triangle T, and A, B, C for the angles. The area of T is its excess $E = A + B + C - \pi$. The old result can be written

$$\tan\frac{E}{2} = \frac{P}{1+\cos a+\cos b+\cos c}, \tag{2}$$

where

$$P = (1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)^{1/2}. \tag{3}$$

The denominators in (1) and (2) are obviously the same. Euler and Lagrange knew also that their numerator (3) was the volume of the parallelepiped with edges $\mathbf{a},\mathbf{b},\mathbf{c}$. Let us prove this fact, starting from the modern expression for that volume

$$V = |[\mathbf{a},\mathbf{b},\mathbf{c}]| = \pm\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

in terms of components of the unit vectors. We then have by the product theorem for determinants

$$V^2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos c & \cos b \\ \cos c & 1 & \cos a \\ \cos b & \cos a & 1 \end{vmatrix} = 1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c.$$

Thus the equivalence of (1) and (2) is established.

The altitude from A of the parallelepiped with edges $\mathbf{a},\mathbf{b},\mathbf{c}$ is (FIGURE 2)

$$h = \sin b\sin C = \sin c\sin B.$$

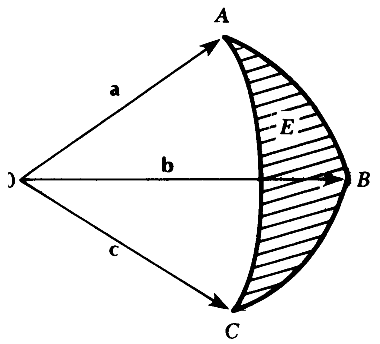

FIGURE 1

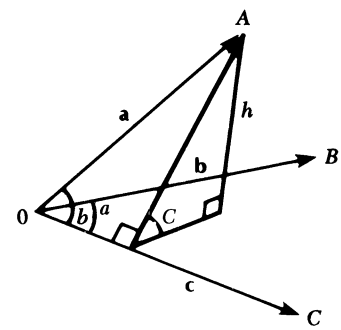

FIGURE 2

This gives, by the way, the Law of Sines:

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}\left(= \frac{\sin a}{\sin A}\right). \tag{4}$$

Since the volume = (area of base) · altitude, we have

$$P = V = \sin a \sin b \sin C. \tag{5}$$

Proofs of (2) can be found in [4], [8] and Todhunter-Leathem [9, p. 104, eq. (30) and p. 28, eq. (17)]. I was not able to find a simpler proof until the referee called my attention to an old theorem, which gives *one* angle representing $(1/2)(\pi - E)$. This theorem [9, p. 116, eq. (20)] is obtained from the following construction (FIGURE 3). Let L and M be the midpoints of BC and AC, respectively, in our spherical triangle ABC. The great circles AB and LM intersect in antipodal points P and Q. Draw arcs AA', BB' and CC' perpendicular to LM. Then triangles AMA' and CMC' are congruent, as are BLB' and CLC'. Thus the angle[1] $(A, MA') = (C, MC') = u$, say, and the angle $(B, B'L) = (C, C'L) = v$. Furthermore $AA' = CC' = BB'$. Thus triangles $AA'P$ and $BB'Q$ are congruent, because the angles at P and Q are equal and the angles at A' and B' are right angles. Hence angle $(A, A'P) = (B, QB')$. Call this angle t. We have now (FIGURE 3):

$$A + u + t = \pi, B + v + t = \pi, C = u + v,$$

and by addition

$$A + B + C + 2t = 2\pi$$

or

$$2t = 2\pi - (A + B + C) = \pi - E.$$

[1] It is convenient to denote the signed angle at A directed from AM to AA' by (A, MA'). In some cases with obtuse angle A, u is to be considered negative, and likewise for v and t (below).

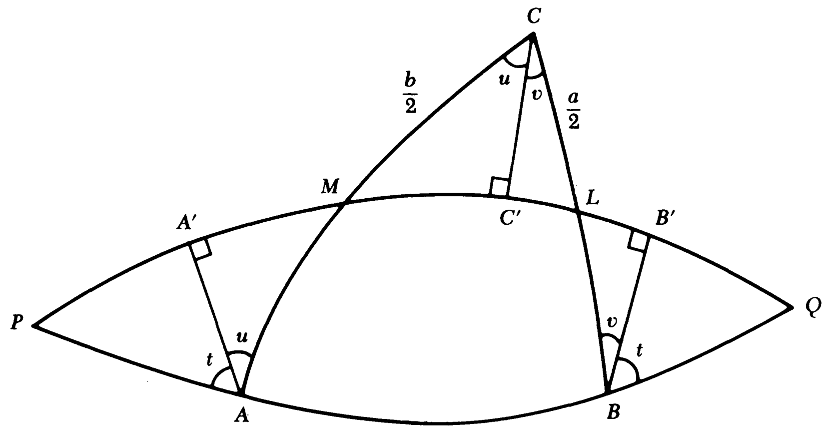

FIGURE 3

Thus

$$\frac{1}{2}(\pi - E) = t = (A, A'P), \tag{6}$$

which is the theorem[2] we need.

Proof of (2). From (6) and FIGURE 3 we get now by elementary spherical trigonometry the following expressions for $\tan E/2$:

$$\tan\frac{E}{2} = \frac{1}{\tan t} = \frac{\sin AA'}{\tan PA'} = \frac{\sin AA'}{\cot ML} = \frac{\sin CC' \sin ML}{\cos ML}$$
$$= \frac{\sin\frac{b}{2}\sin M \sin ML}{\cos ML} = \frac{\sin\frac{b}{2}\sin C \sin\frac{a}{2}}{\cos ML}. \tag{7}$$

(In the second equality we expressed $\tan t$ by a well-known formula for the right-angled triangle $AA'P$. The third equality comes from $PA' + ML = (1/2)PQ = \pi/2$, and the last from the Law of Sines for triangle CML.)

For the denominator $\cos ML$ we get by the Law of Cosines for triangle CML (and in the third equality for ABC)

$$\cos ML = \cos\frac{a}{2}\cos\frac{b}{2} + \cos C \sin\frac{a}{2}\sin\frac{b}{2}$$
$$= \frac{4\cos^2\frac{a}{2}\cos^2\frac{b}{2} + \cos C \sin a \sin b}{4\cos\frac{a}{2}\cos\frac{b}{2}}$$
$$= \frac{(1+\cos a)(1+\cos b) + \cos c - \cos a \cos b}{4\cos\frac{a}{2}\cos\frac{b}{2}}$$
$$= \frac{1 + \cos a + \cos b + \cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}}.$$

[2] I don't know the origin of this theorem. Since [9] gives no reference, and the result is not found in earlier editions of the book (by Todhunter alone), I might guess that it is due to Leathem himself.

Inserting this in the last member of (7) we get

$$\tan\frac{E}{2} = \frac{\sin a \sin b \sin C}{1 + \cos a + \cos b + \cos c},$$

which is (2) with the numerator in the form (5).

Remark 1. The importance of the quantity P was recognized by Lagrange in [**8**]. P, now called the *polar sine* of the solid angle, and its "dual," the 3-*dimensional sine*

$$S = {}^3\sin T = \sin a \sin B \sin C$$

were extensively studied by G. Junghann [**6**, **7**] around 1860. The most striking result in Junghann's "tetrahedrometry" is the Law of Sines: *The areas of the faces of a tetrahedron are proportional to the* 3-*dimensional sines of the opposite corners*. This law was given before in 1850 by Joachimsthal [**5**, p. 40]. It was examined in this MAGAZINE in 1965 by Allendoerfer [**1**]. Compare also [**3**]. The "dual" result of (2) is

$$\tan\frac{a+b+c}{2} = \frac{S}{\cos A + \cos B + \cos C - 1}.$$

Remark 2. Another simple formula for the excess E of the spherical triangle T, given in Keogh's theorem [**9**, p. 119], is worth restating here:

$$\sin\frac{E}{2} = \operatorname{polsin} T',$$

where the right member is the polar sine of the triangle $T' = LMN$ with the midpoints of the sides of T as vertices.

Remark 3. The radii R and r of the circumscribed and inscribed circles of the spherical triangle T can be expressed very simply in terms of our S and P:

$$\tan R = \frac{2}{S}\sin\frac{E}{2}, \quad \cot r = \frac{2}{P}\sin\frac{a+b+c}{2},$$

as in Coxeter's book [**2**, p. 236f].

REFERENCES

1. C. Allendoerfer, Generalizations of theorems about triangles, this MAGAZINE 38 (1965), 253–259.
2. H. S. M. Coxeter, *Non-Euclidean Geometry*, Toronto, 1942.
3. F. Eriksson, The law of sines for tetrahedra and n-simplices, *Geom. Dedicata* 7 (1978), 71–80.
4. L. Euler, De mensura angulorum solidorum, in *Opera omnia* 26, 204–223. (Orig. in *Acta acad. sc. Petrop*. 1778.)
5. F. Joachimsthal, Sur quelques applications des déterminants à la Géométrie, *J. Math*. (Crelle) 40 (1850), 21–47.
6. G. Junghann, Beiträge zur Tetraedrometrie, *Arch. Math. Phys*. (Grunert) 34 (1860), 369–396.
7. ______, *Tetraedrometrie I-II*, Gotha, 1862–63.
8. J.-L. Lagrange, Solutions de quelques problèmes relatifs au triangles sphériques, in *Oeuvres* 7, 331–359. (Orig. in *J. de l'Ecole Polyt*. 1798.)
9. I. Todhunter and J. G. Leathem, *Spherical Trigonometry*, Macmillan, London, 1901.