Exclusive Disjunction and the Biconditional:
An Even-Odd Relationship

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An elementary truth table argument shows that exclusive disjunction is just the negation of the biconditional: \((P \oplus Q) \equiv \neg(P \equiv Q)\). This relationship is sometimes used to explain why inclusive, rather than exclusive, disjunction is the standard disjunction. Either disjunction can be formed from the other \(((P \lor Q) \equiv ((P \oplus Q) \lor (P \land Q))); (P \oplus Q) \equiv ((P \lor Q) \land \neg(P \land Q)))\), but only exclusive disjunction is the negation of another simple connective.

However, while \(P \oplus Q\) is logically equivalent to the negation of \(P \equiv Q\), \(P \oplus Q \equiv R\) is logically equivalent to \(P \equiv Q \equiv R\) itself. (One can omit all parentheses in logical expressions involving only \(\oplus\) or \(\equiv\), since both connectives are commutative and associative.) The reason for this is that \(\oplus\) is a mutual exclusivity connective, whereas \(\equiv\) is an identity connective. Hence, \(P \oplus Q \equiv R\) is true precisely when \(P \equiv Q\) and \(R\) have opposite truth values, which occurs precisely when \(P \equiv Q\) and \(R\) have identical truth values. Generalizing this pattern gives strings of propositions connected by \(\oplus\) or \(\equiv\) that alternate in accordance with the following identities:

\[
\begin{align*}
(A) \quad \bigoplus_{i=1}^{n} P_i &\equiv \left( \bigoplus_{i=1}^{n} P_i \right), \text{ for } n \text{ odd;} \\
(B) \quad \bigoplus_{i=1}^{n} P_i &\equiv \neg\left( \bigoplus_{i=1}^{n} P_i \right), \text{ for } n \text{ even.}
\end{align*}
\]

We now prove these identities by mathematical induction on the number of propositions.

**Proof.** Basis: The logical equivalence \(P_1 \oplus P_2 \equiv \neg(P_1 \equiv P_2)\) follows directly from the truth tables for the two expressions.

Induction Step: Assume the identities true for an integer \(n \geq 2\). We will show them true for \(n + 1\).

(A) \(n\) is odd. We begin with \(\bigoplus_{i=1}^{n+1} P_i\), which can be rewritten \((\bigoplus_{i=1}^{n} P_i) \equiv P_{n+1}\). By the basis, this is equivalent to \(\neg((\bigoplus_{i=1}^{n} P_i) \equiv P_{n+1})\). By the induction hypothesis, this is equivalent to \(\neg((\iff_{i=1}^{n} P_i) \equiv P_{n+1})\). This, in turn, is just \(\neg((\iff_{i=1}^{n+1} P_i)\). This concludes the induction step for case (A) and with it the proof of case (A).

(B) \(n\) is even. We begin with \(\bigoplus_{i=1}^{n+1} P_i\), which can be rewritten \((\bigoplus_{i=1}^{n} P_i) \equiv P_{n+1}\). By the basis, this is equivalent to \(\neg((\bigoplus_{i=1}^{n} P_i) \equiv P_{n+1})\). By the induction hypothesis, this is equivalent to \(\neg((\iff_{i=1}^{n} P_i) \equiv P_{n+1})\). Since \(\neg(P \iff Q)\) is true just when \(P\) and \(Q\) have identical truth values (i.e., \(\neg(P \iff Q) \equiv (P \iff Q)\)), this in turn yields \((\iff_{i=1}^{n+1} P_i) \equiv P_{n+1}\), which is just \(\iff_{i=1}^{n+1} P_i\). This concludes the induction step for case (B) and with it the proof of case (B).

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