## Apropos Predetermined Determinants

Antal E. Fekete, Memorial University of Newfoundland, St. John's, Canada A1C 5S7

In "Predetermined Determinants" [CMJ 16 (September 1985) 227–229], David Buchtal illustrated how determinants whose terms are in arithmetic progression can be used to motivate students to study determinants and their properties. In this capsule, we extend these results to geometric progressions and to arithmetic progressions of higher order.

The value of a determinant whose terms are in geometric progression is zero, because the rows of the determinant are proportional. Thus,

$$\begin{vmatrix} 0.0625 & 0.125 & 0.25 \\ 0.5 & 1 & 2 \\ 4 & 8 & 16 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} 0.0016 & 0.008 & 0.04 \\ 0.2 & 1 & 5 \\ 25 & 125 & 625 \end{vmatrix} = 0.$$

Recall that

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0.$$

One can also compute that

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 5^2 & 6^2 & 7^2 & 8^2 \\ 9^2 & 10^2 & 11^2 & 12^2 \\ 13^2 & 14^2 & 15^2 & 16^2 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} 1^3 & 2^3 & 3^3 & 4^3 & 5^3 \\ 6^3 & 7^3 & 8^3 & 9^3 & 10^3 \\ 11^3 & 12^3 & 13^3 & 14^3 & 15^3 \\ 16^3 & 17^3 & 18^3 & 19^3 & 20^3 \\ 21^3 & 22^3 & 23^3 & 24^3 & 25^3 \end{vmatrix} = 0.$$

In fact, this follows from the more general cases

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = 0$$

and

$$\begin{vmatrix} a^3 & (a+1)^3 & (a+2)^3 & (a+3)^3 & (a+4)^3 \\ b^3 & (b+1)^3 & (b+2)^3 & (b+3)^3 & (b+4)^3 \\ c^3 & (c+1)^3 & (c+2)^3 & (c+3)^3 & (c+4)^3 \\ d^3 & (d+1)^3 & (d+2)^3 & (d+3)^3 & (d+4)^3 \\ e^3 & (e+1)^3 & (e+2)^3 & (e+3)^3 & (e+4)^3 \end{vmatrix} = 0.$$

We can prove this for the $4 \times 4$ determinant of squares by proceeding as follows: reduce the second, third, and fourth columns to first degree polynomials by adding suitable multiples of the first column to each, then reduce the third and fourth columns to constants by adding a suitable multiple of the second column to each, then reduce the fourth column to zeros by adding a suitable multiple of the third column to the fourth column. For the $5 \times 5$ determinant of cubes, this type of procedure reduces the fifth column to zeros.

These results can be further generalized to determinants whose entries are members of an arithmetic progression of higher order. For a given sequence $a_1, a_2, a_3, \ldots$ the sequence of differences of consecutive terms $\Delta a_1 = a_2 - a_1$, $\Delta a_2 = a_3 - a_2$, $\Delta a_3 = a_4 - a_3, \ldots$ is called the first-order difference sequence; higher order difference sequences are formed by repeating this procedure on the preceding difference sequence. An arithmetic progression of order $k$ is a sequence for which the $k$th order difference sequence is the last one that does not vanish. In other words, the $k$th order difference sequence is the constant sequence $d, d, d, \ldots$ with $d \neq 0$. Thus,

$$\begin{array}{ccccccccc} 1 & & 4 & & 9 & & 16 & & 25\ldots \\ & 3 & & 5 & & 7 & & 9 & \ldots \\ & & 2 & & 2 & & 2 & & \ldots \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} 1 & & 8 & & 27 & & 64 & & 125\ldots \\ & 7 & & 19 & & 37 & & 61 & \ldots \\ & & 12 & & 18 & & 24 & \ldots & \\ & & & 6 & & 6 & \ldots & & \end{array}$$

show that the consecutive squares (cubes) form an arithmetic sequence of order 2 (order 3). More generally, as is well known, the consecutive $k$th powers form an arithmetic sequence of order $k$. [See Calvin Long's "Pascal's Triangle, Difference Tables, and Arithmetic Sequences of Order $N$," CMJ 15 (September 1984) 290–298.]

Further examples are the polygonal and pyramidal numbers. The triangular numbers $(\frac{1}{2})n(n-1)$, the pentagonal numbers $(\frac{1}{2})n(3n-1)$, the hexagonal numbers $n(2n-1)$, etc., form arithmetic progressions of order two; the tetrahedral numbers $\binom{n}{3}$ form an arithmetic progression of order three.

Our main result is the following.

**Theorem.** *Let $a_1, a_2, a_3, \ldots$ be an arithmetic progression of order $k$, and let $d$ be the constant obtained as the $k$th difference sequence. Then*

$$D = \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_{n+1} & a_{n+2} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & a_{n^2} \end{vmatrix} = \begin{cases} 0, & k \leqq n-2 \\ (-n)^{\binom{n}{2}} d^n, & k = n-1 \end{cases}.$$

To prove this, we begin by subtracting the first column from the second, the second column from the third, etc. In this way, we obtain a determinant whose second, third, ..., $n$th columns contain the elements of the first difference sequence. We continue by subtracting the second column from the third, the third column from the fourth, etc., to obtain a determinant whose third, fourth, ..., $n$th columns contain the elements of the second difference sequence. In the $(n-1)$st step, we get a determinant whose last column consists of the elements of the $(n-1)$st difference sequence. If $k \leqq n-2$ (equivalently, $n-1 \geqq k+1$), then the last column is 0, and hence the determinant $D = 0$.

If $k \geqq n-1$, then $D \neq 0$ and one may think that the determinant is no longer "predetermined." Therefore, it may come as a surprise that in the case $k = n-1$, the value of the determinant does not depend on the actual members of the arithmetic progression, so long as consecutive members are entered. For example:

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 4^2 & 5^2 & 6^2 \\ 7^2 & 8^2 & 9^2 \end{vmatrix} = \begin{vmatrix} 2^2 & 3^2 & 4^2 \\ 5^2 & 6^2 & 7^2 \\ 8^2 & 9^2 & 10^2 \end{vmatrix} = \begin{vmatrix} 3^2 & 4^2 & 5^2 \\ 6^2 & 7^2 & 8^2 \\ 9^2 & 10^2 & 11^2 \end{vmatrix} = \cdots = -216.$$

This is a consequence of the following result, which may also be of interest in its own right.

**Lemma.** *Let $a_0, a_1, a_2, \ldots$ be an arithmetic progression of order k with $\Delta^k a_n = d$ for $n = 0, 1, 2, \ldots$. Then $a_0, a_m, a_{2m}, \ldots$ is also an arithmetic progression of order k with $\Delta^k a_{mn} = m^k d$ for $n = 0, 1, 2, \ldots$. More generally, a subsequence formed by taking every mth member, starting with an arbitrary member of the arithmetic progression of order k, is also an arithmetic progression of order k.*

*Proof of the Lemma.* An arithmetic progression of order $k$ can be expressed as a polynomial of degree $k$:

$$a_n = c_k n^k + c_{k-1} n^{k-1} + \cdots + c_0 \qquad (c_k \neq 0). \tag{$*$}$$

In more detail, $a_n = (1+\Delta)^n a_0 = \sum_{j=0}^{k} \binom{n}{j} \Delta^j a_0$ because $\Delta^{k+1} a_0 = \cdots = \Delta^n a_0 = 0$. Since the $\Delta^j a_0$ are constants and $\binom{n}{k}$ is a polynomial of degree $k$, we express $a_n$ as in ($*$). By linearity of the operator $\Delta$, and because $\Delta^k n^h = 0$ for $h \leqq k-1$, we have $\Delta^k a_n = \Delta^k c_k n^k = c_k \Delta^k n^k = d$ for $n = 0, 1, 2, \ldots$. Therefore, for

$$a_{mn} = c_k (mn)^k + c_{k-1} (mn)^{k-1} + \cdots + c_0,$$

we have

$$\Delta^k a_{mn} = \Delta^k c_k m^k n^k = m^k c_k \Delta^k n^k = m^k d \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

The general case follows from this, because we may omit the first $N$ members of the arithmetic progression of order $k$, the remainder is an arithmetic progression of the same order.

We may now conclude the proof of the Theorem as follows. In order to calculate the value of $D$ in the case $k = n - 1$, observe that the steps prescribed in the first part of the proof have brought the original determinant to the form

$$D = \begin{vmatrix} a_1 & \Delta a_2 & \cdots & \Delta^{n-2} a_{n-1} & \Delta^{n-1} a_n \\ a_{n+1} & \Delta a_{n+2} & \cdots & \Delta^{n-2} a_{2n-1} & \Delta^{n-1} a_{2n} \\ a_{2n+1} & \Delta a_{2n+2} & \cdots & \Delta^{n-2} a_{3n-1} & \Delta^{n-1} a_{3n} \\ a_{3n+1} & \Delta a_{3n+2} & \cdots & \Delta^{n-2} a_{4n-1} & \Delta^{n-1} a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}.$$

By the Lemma, the elements in the $j$th column are in arithmetic progression of order $n - j$. In particular, each element in the last column is the constant $d$. Moreover, the $j$th difference sequence of the elements in the $(n-j)$th column is constant and is equal to $n^j d$.

Therefore, we may bring the determinant to triangular form as follows. Subtract the first row from the second, the second row from the third, etc., making all the elements except the first one in the $n$th column equal to zero, and all the elements except the first one in the $(n-1)$st column equal to the constant value $nd$. Then subtract the second row from the third, the third row from the fourth, etc., making all the elements except the first two in the $(n-1)$st column equal to zero, and all the elements except the first two in the $(n-2)$nd column equal to the constant $n^2 d$. Proceeding in this way, after the $(n-1)$st step we get

$$D = \begin{vmatrix} * & * & \cdots & * & * & d \\ * & * & \cdots & * & nd & 0 \\ * & * & \cdots & n^2d & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ n^{n-1}d & 0 & \cdots & 0 & 0 & 0 \end{vmatrix} = (-1)^{\binom{n}{2}} \begin{vmatrix} n^{n-1}d & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & \cdots & n^2d & 0 & 0 \\ * & * & \cdots & * & nd & 0 \\ * & * & \cdots & * & * & d \end{vmatrix}$$

$$= (-1)^{\binom{n}{2}} n^{1+2+\cdots+n-1} d^n = (-n)^{\binom{n}{2}} d^n.$$

As the signature indicates, it takes $(n-1)+(n-2)+\cdots+2+1=\binom{n}{2}$ switches of rows to bring the determinant to the form in which the upper right triangle consists of zeros. As the students will recall, the value of such a determinant can be calculated by taking the product of the elements in the main diagonal.

*Acknowledgement.* The author wishes to express his gratitude to the referees for their helpful suggestions.

---o---

## An Efficient Logarithm Algorithm for Calculators

James C. Kirby, Tarleton State University, Stephenville, TX

Dan Kalman and Warren Page [CMJ 16 (January 1985) 57–60] discussed algorithms for raising a number to large or nonintegral powers on calculators equipped with square and square-root keys but having no general power or root key. In this capsule, we will present an algorithm for computing natural logarithms on calculators not equipped with a logarithm key. Initially, we assume that the only keys available are $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, $\boxed{M+}$, $\boxed{M-}$, and $\boxed{MR}$, where the latter three access one memory cell. We will then examine how the algorithm is improved if there is a square-root key. Our intent, as was Kalman and Page's, is to use this as a classroom discussion which combines mathematical applications and algorithmic reasoning.

Integrating the expansion

$$\frac{1}{(1+x)} = 1 - x + x^2 - x^3 + \cdots \qquad |x|<1,$$

we obtain

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots \qquad |x|<1. \tag{1}$$

There are two problems with this expansion: it has a small radius of convergence, and it converges very slowly. To obtain a series which converges more rapidly, replace $x$ with $-x$, subtract the two expressions, and rewrite the result as

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left[x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \cdots\right] \qquad |x|<1.$$

Now if we let $N \in Z^+$ and set $x = 1/(2N+1)$, we obtain

$$\ln\left(1+\frac{1}{N}\right) = 2\left[\frac{1}{2N+1} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \cdots\right].$$

Although this expansion only works for natural numbers $N$, it can be written recursively as

$$\ln(N+1) = \ln N + \Delta N, \tag{2}$$

where $\Delta N$ is the right-hand series in the above equation. This motivates us to