Why Are the Gergonne and Soddy Lines Perpendicular? A Synthetic Approach

ZUMING FENG
Phillips Exeter Academy
Exeter, NH 03833
zfeng@exeter.edu

In any scalene triangle the three points of tangency of the incircle together with the three vertices can be used to define three new points which are, remarkably, always collinear. This line is called the Gergonne Line. Moreover cevians through these tangent points are always concurrent at a common point that, together with the incenter, defines a second line, the Soddy Line. Why should these lines be perpendicular? Beauregard and Suryanarayan [1] used Euclidean coordinates to establish these results, whereas Oldknow [2] used trilinear coordinates. But such a geometric gem deserves a synthetic geometric proof. We shall use the classical theorems of Ceva and Menelaus to define these lines and then establish their perpendicularity by using a certain inversion.

Let $A_C B_C$ be a scalene triangle. Let $\Omega$ and $I$ be its incircle and incenter, respectively. Circle $\Omega$ touches sides $A B, B C, C A$ at $C_1, A_1, B_1$, respectively. Lines $A_1 B_1$ and $A B$ meet at $C_2$, and points $A_2$ and $B_2$ are defined analogously. As usual, we set $A B = c, B C = a, C A = b$, and $s = \frac{a+b+c}{2}$. Then it is well known that $A B_1 = A C_1 = s - a, B A_1 = B C_1 = s - b,$ and $C A_1 = C B_1 = s - c.$

The Gergonne Line. A lovely result for identifying collinear points is Menelaus’ theorem:

Let $A C B$ be a triangle, and let $P, Q, R$ be points on the lines $B C, C A, A B$, respectively. Then $P, Q, R$ are collinear if and only if

$$\frac{B P}{P C} \cdot \frac{C Q}{Q A} \cdot \frac{A R}{R B} = 1.$$  

(If the lengths are directed, then the product is $-1$.)

Applying Menelaus’ Theorem to line $B_1 C_1$ with triangle $A B C$ (FIGURE 1) yields

$$1 = \frac{A C_1}{C_1 B} \cdot \frac{B A_2}{A_2 C} \cdot \frac{C B_1}{B_1 A} = \frac{B A_2 \cdot (s - a)(s - c)}{A_2 C \cdot (s - b)(s - a),}$$

and so

$$\frac{B A_2}{A_2 C} = \frac{s - b}{s - a}.$$
Likewise, we have
\[
\frac{CB_2}{B_2A} = \frac{s - c}{s - b} \quad \text{and} \quad \frac{AC_2}{C_2B} = \frac{s - a}{s - c}.
\]
Consequently,
\[
\frac{BA_2}{A_2C} \cdot \frac{CB_2}{B_2A} \cdot \frac{AC_2}{C_2B} = 1,
\]
implying that \(A_2, B_2, C_2\) are collinear, by Menelaus' Theorem. Thus, the line passing through points \(A_2, B_2,\) and \(C_2\) is uniquely defined (FIGURE 1). This is the Gergonne Line of triangle \(ABC\).

**The Soddy Line.** A cevian of a triangle is any segment joining a vertex to a point on the opposite side. We can test cevians for concurrence by using Ceva’s Theorem:

Let \(AD, BE, CF\) be three cevians of triangle \(ABC\). Then segments \(AD, BE, CF\) are concurrent if and only if
\[
\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.
\]

Note that
\[
\frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B} \cdot \frac{BC_1}{C_1A} = \frac{(s - a)(s - c)(s - b)}{(s - c)(s - b)(s - a)} = 1.
\]
By Ceva’s Theorem, lines \(AA_1, BB_1,\) and \(CC_1\) are concurrent (FIGURE 1). The point of concurrency is the Gergonne point \(Ge\) of triangle \(ABC\). Because our triangle is scalene, \(I\) does not lie on any of the lines \(AA_1, BB_1,\) and \(CC_1\). Hence \(I\) and \(Ge\) are distinct points. Thus there is a unique line passing through \(I\) and \(Ge\) (FIGURE 1). This line is the Soddy line of triangle \(ABC\).

**The Gergonne line and the Soddy line are perpendicular.** We apply a certain inversion to show that \(A_2B_2 \perp IG_e\). Given a point \(O\) in the plane and a real number \(r > 0\), the inversion through \(O\) with radius \(r\) maps every point \(P\) (distinct from \(O\)) to the point \(P'\) on the ray \(OP\) such that \(OP \cdot OP' = r^2\). We also refer to this map as inversion through \(\gamma\), the circle with center \(O\) and radius \(r\). Key properties of inversion that will be used are (for details, please see [3]):

(a) Lines passing through \(O\) invert to themselves (though the individual points on the line are not all fixed, FIGURE 2, left).

(b) Lines not passing through \(O\) invert to circles through \(O\), and vice versa (FIGURE 2, middle).
(c) Circles not passing through $O$ invert to circles not passing through $O$ (Figure 2, right).

(d) Inversion is a conformal map; that is, inversion preserves the angle between (the tangent lines of) any curves at their intersection points.

We consider the inversion $I$ with respect to the incircle $\Omega$ (Figure 3). Let $I(P)$ denote the image of element $P$ under the inversion. Then $I(A_1) = A_1$, $I(B_1) = B_1$, and $I(C_1) = C_1$. Because $\angle IB_1A = \angle IC_1A = \frac{\pi}{2}$, points $A$, $B_1$, $I$, $C_1$ lie on a circle. Let $\Omega_a$ denote this circle. We define circles $\Omega_b$ and $\Omega_c$ (Figure 3) analogously. By property (b), $I(\Omega_a) = B_1C_1$, $I(\Omega_b) = C_1A_1$, and $I(\Omega_c) = A_1B_1$.

Let $\Omega_1$ be the circle with segment $IA_1$ as a diameter (Figure 4). Then the image of $\Omega_1$ under the inversion is a line, by property (b). Because $\Omega$ and $\Omega_1$ are tangent to each other at $A_1$, by property (d), their images should also be tangent to each other at the image of $A_1$. It follows that $I(\Omega_1) = BC$. 

Figure 2

Figure 3

Figure 4
Let $A_3$ be the foot of the perpendicular from $I$ to segment $AA_1$ (Figure 4). Because $AI$ is a diameter of $\Omega_a$ and $IA_1$ is a diameter of $\Omega_1$, $A_3$ lies on both $\Omega_a$ and $\Omega_1$. Thus, $I(A_3)$ is the intersection of $I(\Omega_a) = B_1C_1$ and $I(\Omega_1) = BC$; that is, $I(A_3) = A_2$ (Figure 1).

![Figure 5](image)

Points $B_3$ and $C_3$ are defined analogously and the equations $I(B_3) = B_2$ and $I(C_3) = C_2$ follow in similar manner (Figure 5). Because $\angle IA_3 Ge = \angle IB_3 Ge = \angle IC_3 Ge = \frac{\pi}{2}$, points $A_3$, $B_3$, $C_3$, $I$, $Ge$ lie on a circle $\Gamma$ with $IGe$ as its diameter. By property (b), $I(\Gamma)$ is a line; that is, points $A_2$, $B_2$, $C_2$ lie on a line. (This is another proof of the existence of the Gergonne line. Furthermore, this shows that $I(Ge)$ also lies on the Gergonne line.) By property (a), the image of ray $IGe$ is ray $IGe$. By property (d), to show $A_2B_2 \perp IGe$ (Figure 1), it suffices to show that circle $\Gamma$ and ray $IGe$ are perpendicular at their intersection point $Ge$. But this is evident, because $IGe$ is a diameter of circle $\Gamma$ (Figure 5).

REFERENCES


Golden Matrix Ring Mod $p$

KUNG-WEI YANG
Western Michigan University (Retired)
Kalamazoo, MI 49008

Playing with the “golden matrix ring” $\mathbb{Z}[A]$ ($A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$), we had fun proving identities involving Fibonacci numbers ($F_0 = 0$, $F_1 = 1$, 1, 2, 3, 5, 8, 13, 21, ... in [6]). Here we return to $\mathbb{Z}[A]$ and show that if we reduce $\mathbb{Z}[A]$ modulo $p$, then we will get a very neat proof of one of the more remarkable properties of Fibonacci numbers: every prime $p$ divides some (hence infinitely many) Fibonacci numbers.