Fibonacci Numbers and the Arctangent Function

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This note provides several geometric illustrations of three identities involving the arctangent function and the reciprocals of Fibonacci numbers. The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n > 1$. The following identities link the Fibonacci numbers to the arctangent function. Only the first is evident in the literature [**1, 2, 3**].

$$\arctan\left(\frac{1}{F_{2i}}\right) = \arctan\left(\frac{1}{F_{2i+1}}\right) + \arctan\left(\frac{1}{F_{2i+2}}\right) \tag{1}$$

$$\arctan\left(\frac{2}{F_{2i+2}}\right) = \arctan\left(\frac{1}{F_{2i+1}}\right) + \arctan\left(\frac{1}{F_{2i+4}}\right) \tag{2}$$

$$\arctan\left(\frac{1}{F_{2i}}\right) = \arctan\left(\frac{2}{F_{2i+2}}\right) + \arctan\left(\frac{1}{F_{2i+3}}\right) \tag{3}$$

Identities (1)–(3) can be proven formally using Cassini's identity [**1**, p. 127]

$$F_{k+1}^2 = F_k F_{k+2} + (-1)^k$$

and the addition formula for the tangent function. Interested readers are invited to do so.

The following six diagrams illustrate special cases of equations (1)–(3). FIGURE 1, a representation of Euler's famous formula for π [**4, 5**], illustrates (1) for $i = 1$. One can see that $\angle ABD$ plus $\angle DBC$ is equal to $\angle ABC$.

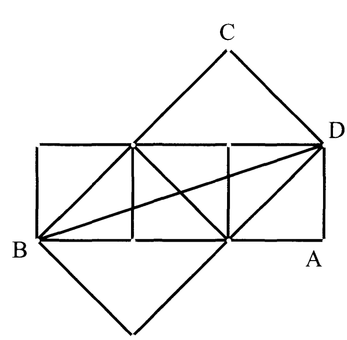

Figure 1 $\frac{\pi}{4} = \arctan(1) = \arctan(\frac{1}{2}) + \arctan(\frac{1}{3})$

FIGURE 2 illustrates (1) for $i = 2$, using the larger squares to form the arctangent of $1/5$ and the smaller squares being used to form the arctangents of $1/3$ and of $1/8$.

The two diagrams in FIGURE 3 illustrate (2) for the values $i = 1$ and $i = 2$.

The diagrams in FIGURE 4 illustrate equation (3) for the values $i = 1$ and $i = 2$.

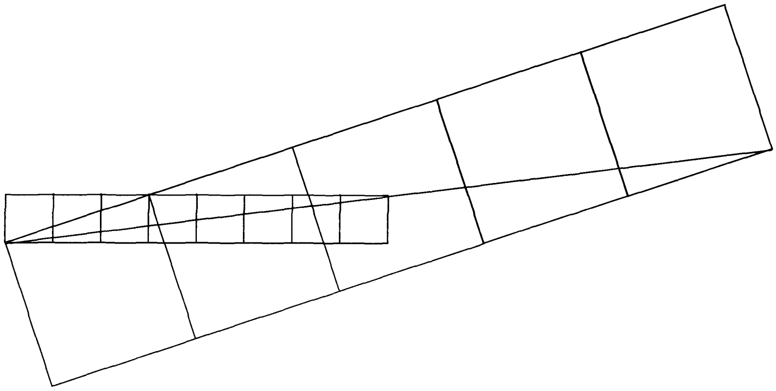

Figure 2 $\arctan(\frac{1}{3}) = \arctan(\frac{1}{5}) + \arctan(\frac{1}{8})$

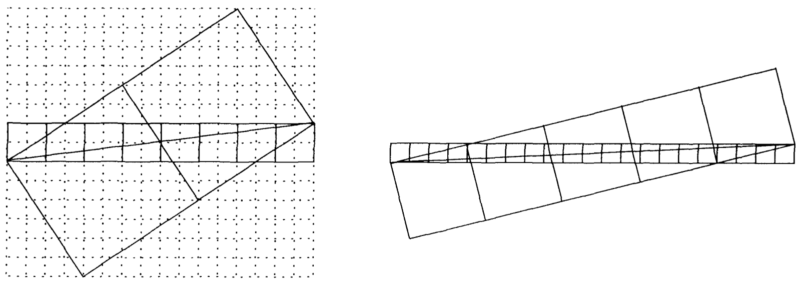

Figure 3 $\arctan(\frac{2}{3}) = \arctan(\frac{1}{2}) + \arctan(\frac{1}{8})$; $\arctan(\frac{1}{4}) = \arctan(\frac{1}{5}) + \arctan(\frac{1}{21})$

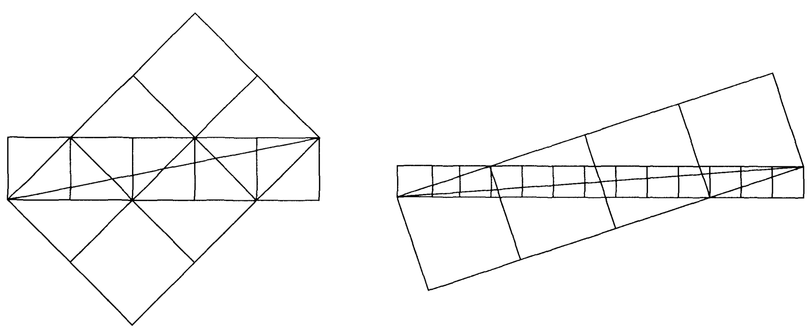

Figure 4 $\arctan(1) = \arctan(\frac{2}{3}) + \arctan(\frac{1}{5})$; $\arctan(\frac{1}{3}) = \arctan(\frac{1}{4}) + \arctan(\frac{1}{13}))$

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