On the Symmetry Group of the Dodecahedron

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1. Introduction In this note we offer a proof of a well-known theorem on the group of symmetries of the dodecahedron, and thus of its dual, the icosahedron, which involves only the most easily visualized features of the solid, employing arguments from elementary group theory. This approach differs from that of the previous commonly-known proofs which entail visualization of (embedded or projected) supplementary figures associated with these solids (cf., [1, p. 47]).

We let $P$ denote a dodecahedron with center $O$ and let $\lambda$ denote the (nonrealizable) symmetry taking each point $T$ of $P$ to the point symmetric to $T$ in $O$. Pairs of opposite faces \{\( F, \lambda(F) \)\} and a distinguished face $F$ and vertex $V$ are labeled as shown in Figures 1 and 2. (Visible faces have boldface numbers.) We let $G$ denote the group of symmetries of $P$ which are realizable (by motions) in three-space. In §2, using arguments from elementary group theory that do not involve Sylow theory, we prove:

**Theorem A.** $G \cong A_5$ (the alternating group on 5 letters).

In §3, for completeness, we identify the full group $G'$ of symmetries of $P$.

**FIGURE 1**

**FIGURE 2**
2. Proof of Theorem A We first observe that \(|G| = 60\). For, given any face \(F'\) of the 12 faces of \(P\) and any vertex \(V'\) of the 5 vertices of \(F'\), it is clearly possible to move \(P\) around \(O\) in such a fashion that \(F\) and \(V\) occupy the former positions of \(F'\) and \(V'\). (The positions of the remaining vertices of \(F\), and hence of \(P\) are thus completely determined since orientation—clockwise or counterclockwise—is preserved on the surface of \(P\).) Further, we note that each \(\sigma_p \in G\) determines a permutation \(\sigma\) of the six pairs of faces of \(P\), and that each such permutation determines at most one \(\sigma_p \in G\), as the vertex \(V\) is surrounded by faces labeled 1, 2 and 3 in clockwise order, and \(V\) is on \(F\). We conclude that \(G\) (a group under composition of mappings) is isomorphic to a subgroup \(H\) of \(S = S_6\), the symmetric group on 6 letters. The rotation \(\alpha_1\) depicted in Figure 1, about the axis joining the centers of \(F\) and \(\lambda(F)\), corresponds to \(\alpha = (1)(2, 3, 4, 5, 6) \in H\). Also, the rotation \(\beta_1\) about \(V\lambda(V)\) depicted in Figure 2 corresponds to \(\beta = (1, 2, 3)(4, 6, 5)\). We observe (multiplying permutations from right to left) that \(\delta = \alpha \beta = (1, 3)(2, 4)(5, 6)\), \(\omega = \beta \alpha^{-1} = (1, 6, 4, 5, 3)(2)\), \(\rho = \alpha^3 \beta^3 = (1, 2, 5)(3, 4, 6)\) and \(\gamma = \rho \delta \rho^{-1} = (1)(3)(2, 4)(5, 6)\) are in \(H\). Since \(\gamma \delta = \delta \gamma\), the group \(\langle \gamma, \delta \rangle\) generated by \(\delta\) and \(\gamma\) is a copy of the Klein four-group. Also, by Lagrange's theorem, \(4||\langle \alpha, \beta \rangle|\). Since \(5 = o(\alpha)\) and \(3 = o(\beta)\), we have \(15 ||\langle \alpha, \beta \rangle|\) also, so that \(60 ||\langle \alpha, \beta \rangle|\) and \(H = \langle \alpha, \beta \rangle \cong G\).

We will show that \(A_5\) is also isomorphic to \(H\). We begin by observing that \(A_5\) contains 24 (= 4!) 5-cycles and thus 6 subgroups \(H_i\) of order 5. For each \(g \in A_5\) we define \((f[g])(i)\) to be \(j\), where \(gh_i g_i^{-1} = H_j\). Clearly \(f[g]\) is one-to-one so that \(f[g] \in S\). Since

\[(g_1 g_2)H_1(g_1 g_2)^{-1} = g_1(g_2 H_i g_2^{-1})g_1^{-1}, (f[g_1 g_2])(i) = f[g_1](f[g_2](i)) \quad \text{for all } i,\]

and thus the mapping \(T: A_5 \to S\) given by \(T(g) = f[g]\) for all \(g \in A_5\) is a homomorphism.

We recall that to compute \(\tau \mu^{-1}\) one replaces every symbol in the disjoint cycle form of \(m\) by its image under \(\tau\). Thus, we can easily choose generators \(i\) for the subgroups \(H_i\) of \(A_5\) in such a manner that: \(\sigma_1 = (1, 2, 3, 4, 5)\), \(\sigma_2 = (1, 2, 3, 5, 4)\), \(\sigma_3 H_i \sigma_1^{-1} = H_{i+1}\) for \(2 \leq i \leq 5\) and \(\sigma_1 H_5 \sigma_1^{-1} = H_2\). (Since \(\sigma_3 \sigma_2 \sigma_1^{-1} = (2, 3, 4, 1, 5) = (1, 2, 4, 5, 3)\), we take \(\sigma_3 = (1, 2, 4, 5, 3)\). Similarly we choose \(\sigma_4 = (1, 2, 5, 3, 4)\) and \(\sigma_5 = (1, 2, 4, 3, 5)\). Thus \(f[\sigma_1] = \lambda(2, 3, 4, 5, 6) = \alpha\). Letting \(\tau_1 = (1)(4)(2, 3, 5)\), we find that \(f[\tau_1] = (1, 2, 3)(4, 6, 5) = \beta\). Hence \(T(A_5) = H\), and since \(|A_5| = 60\), \(A_5 \cong H\), so that \(G \cong A_5\), as asserted. \(\Box\)

3. The full group \(G'\) of symmetries of \(P\) Arguing as in §2, we conclude that \(G'\) contains precisely 60 elements which reverse orientation, and these are just the mappings \(\sigma \lambda\), where \(\sigma \in G\). Since \(\lambda\) leaves the lines joining centers of opposite pairs of faces fixed, \(\sigma \lambda = \lambda \sigma\) for all \(\sigma\). Hence \(G' = G \times \langle \lambda \rangle \cong A_5 \times C_2\) (where \(C_2\) is a cyclic group of order 2) and \(G'\) has order 120. We note that \(G\) contains 24 rotations of order 5 about axes joining centers of opposite faces and 20 rotations of order 3 about axes joining opposite vertices. Also, it contains 15 rotations of measure \(\pi\) about axes joining midpoints \(M, \lambda(M)\) of opposite edges (see Figure 2). Thus all 60 physically realizable symmetries of \(P\) are rotations, and conversely. (Of course, the geometers among us already knew this, as it is true for an arbitrary bounded solid.)

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REFERENCE