Visualizing Leibniz's Rule

MARC FRANTZ
Indiana University
Bloomington, IN 47405

Leibniz's rule for differentiating under the integral sign deals with functions of the form

$$A(x) = \int_0^{\beta(x)} f(x, y)\, dy. \tag{1}$$

If f and $f_x = \partial f/\partial x$ are continuous in a suitable region of the plane, and if β' is continuous over a suitable interval, Leibniz's rule says that A' is continuous, and

$$A'(x) = \int_0^{\beta(x)} f_x(x, y)\, dy + f(x, \beta(x))\beta'(x). \tag{2}$$

(A formal statement of the rule appears at the end of this note.) Most textbooks generalize Leibniz's rule to the case when the lower limit of integration in Equation (1) is also a function $\alpha(x)$. This is easily done, since

$$\int_{\alpha(x)}^{\beta(x)} f(x, y)\, dy = \int_0^{\beta(x)} f(x, y)\, dy - \int_0^{\alpha(x)} f(x, y)\, dy.$$

The reader can find various derivations and applications* of Leibniz's rule in advanced calculus texts (cf. [**2**], [**3**], [**4**]). But what interpretation can we give to the two terms—an integral and a product—on the right side of Equation (2)? An integral is the area under a curve, or between two curves, and a product of two factors can be the area of a rectangle. I will show that this is essentially the correct geometric interpretation, and give an informal derivation of Leibniz's rule which may appeal to visually oriented students.

First, we need a geometric interpretation of $A(x)$, and this is given in FIGURE 1.

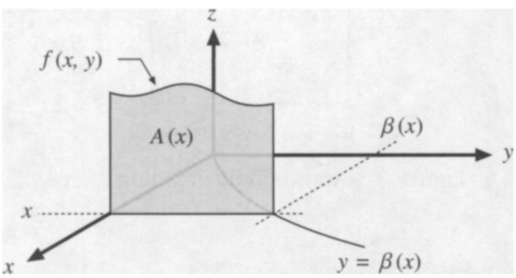

Figure 1 Geometric interpretation of $A(x)$

Here we imagine a surface $z = f(x, y)$ above the xy-plane, and a curve $y = \beta(x)$ lying in the xy-plane itself. If we now consider x fixed, then $A(x)$ as given in Equa-

*A recent application in this MAGAZINE [1] prompted me to think about the rule again.

Mathematical Association of America
is collaborating with JSTOR to digitize, preserve, and extend access to
Mathematics Magazine
www.jstor.org

tion (1) is just the area of the lamina (shaded in FIGURE 1) beneath the curve $z(y) = f(x, y)$.

The next step (FIGURE 2) is to consider a corresponding lamina at a new x-coordinate $x + \Delta x$. I have depicted this lamina, whose area is $A(x + \Delta x)$, as being larger than the first, with the area $A(x)$ of the first lamina projected forward onto it for comparison. With x and Δx fixed, the upper boundary is now the curve $z(y) = f(x + \Delta x, y)$, and at any y-coordinate y, the increase in height is approximately $f_x(x, y)\Delta x$. Similarly, the increase in the lamina's width is approximately $\beta'(x)\Delta x$.

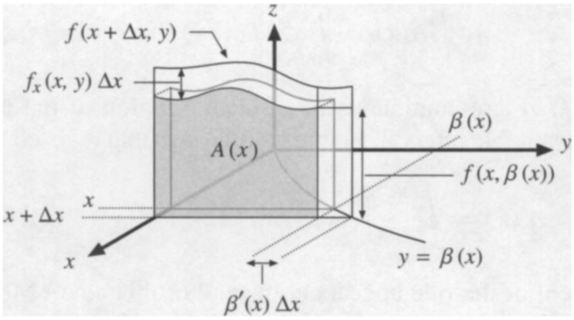

Figure 2 $A(x)$ projected onto $A(x + \Delta x)$

Finally, I've taken the lamina in FIGURE 2 and isolated it in FIGURE 3 for clarity. The area of the small piece in the upper right corner is proportional to $(\Delta x)^2$, so to first order accuracy in Δx we can ignore it. Thus the increase $\Delta A = A(x + \Delta x) - A(x)$ in area is approximately equal to that of the two shaded regions in Figure 3:

$$\Delta A \approx A_1 + A_2. \tag{3}$$

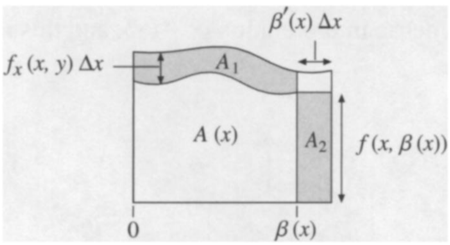

Figure 3 Approximate shading of ΔA

It should be clear from FIGURE 3 that

$$A_1 \approx \int_0^{\beta(x)} f_x(x, y)\, dy\, \Delta x \qquad \text{and} \qquad A_2 \approx f(x, \beta(x))\beta'(x)\Delta x. \tag{4}$$

This is the desired geometric interpretation; the two terms on the right side of Equation (2) are proportional, respectively, to the areas A_1 and A_2 in FIGURE 3. Indeed, we can use Equations (4) to substitute for A_1 and A_2 in Equation (3), and divide by Δx to obtain

$$\frac{\Delta A}{\Delta x} \approx \int_0^{\beta(x)} f_x(x, y)\,dy + f(x, \beta(x))\beta'(x).$$

Letting $\Delta x \to 0$ gives us Leibniz's rule.

Comment This approach seemed so natural to me that I was surprised not to find it in a textbook. However, one of the referees found a closely related exposition in an old classic text [**5**].

Formal statement of Leibniz's rule. Let $A(x)$ be given by (1). If f and $f_x = \partial f/\partial x$ are continuous in a region R, and if $\beta(x) \geq 0$ is continuously differentiable for $a \leq x \leq b$, and if $\{(x, y) \mid a \leq x \leq b,\ 0 \leq y \leq \beta(x)\} \subset R$, then A is continuously differentiable for $a \leq x \leq b$ and $A'(x)$ is given by (2).

Acknowledgment. My thanks to Barbara Nimershiem for pointing out Colin Adams' article [**1**], and for my going-away present—the question that led to this note.

REFERENCES

1. Colin C. Adams, The newest inductee in the number hall of fame, this MAGAZINE **71** (1998), 341–349.
2. F. B. Hildebrand, *Advanced Calculus, Second Ed.*, Prentice-Hall, Englewood Cliffs, NJ, 1976.
3. W. Kaplan, *Advanced Calculus, Second Ed.*, Addison-Wesley, Reading, MA, 1973.
4. M. R. Spiegel, *Advanced Calculus*, Schaum's Outline Series, McGraw-Hill, New York, 1997.
5. E. B. Wilson, *Advanced Calculus*, Ginn & Co., Boston, 1912.

A Celestial Cubic

C. W. GROETSCH
University of Cincinnati
Cincinnati, OH 45221-0025

...Nature is pleased with simplicity and affects not
the pomp of superfluous causes.
Newton

Introduction The Greek philosopher Epicurus (342?–270 BC) espoused an infinity of worlds like our own. Later thinkers followed suit, sometimes with tragic consequences—Giordano Bruno (1548?–1600) was consumed by the flames of the Inquisition for preaching, among other things, a plurality of worlds. In modern times belief in the existence of extra-solar planets has been nearly universal (leading sometimes to comedy, rather than tragedy—viz. UFOs, alien abductions, etc.), yet the immense inter-stellar distances involved have defeated efforts to observe such planets directly. In fact, direct visual observation of extra-solar planets has been compared to naked-eye viewing, at a distance of many miles, of a moth fluttering around a porch light.

Recently convincing *indirect* evidence of planets orbiting specific stars has emerged. In the past indirect evidence has led to major astronomical discoveries. Indirect visual evidence, in the form of observations of orbital perturbations of Uranus, combined with the solution of a difficult mathematical inverse problem by Adams and Leverrier