**INTRODUCTION**

The process of deforming a curve by the curvature vector at each point is known as the curve-shortening flow (CSF). Grayson showed that CSF averages the shape of a curve, causing simple, closed curves to become asymptotically circular. This research applies the idea of curvature flow to polygons.

We give a novel definition for the ‘curvature vector’ at a vertex of a polygon. Deforming in the direction of this curvature vector yields a flow of polygons, the polygon curvature flow (PCF). We numerically investigate the behavior of this flow and show that it exhibits several of the qualitative properties of CSF. We conjecture that PCF makes polygons asymptotically regular.

**CURVE-SHORTENING FLOW**

The curve-shortening flow (CSF) moves each point of a curve $\Gamma$ in the direction of its curvature vector $\vec{\kappa}$ by applying the ‘heat equation’ to that curve

$$\frac{\partial \Gamma}{\partial t} = \frac{\partial^2 \Gamma}{\partial s^2} = \frac{\partial}{\partial s} \left( \frac{\partial \Gamma}{\partial s} \right) = \vec{\kappa},$$

One way to define the curvature vector $\vec{\kappa}$ is

$$\vec{\kappa} = \frac{d\theta}{ds} \vec{\nu},$$

where $\vec{\nu}$ is the inward pointing normal, and $d\theta/ds$ is the change in angle between the tangent vector and any reference vector.

**REGULARITY AND END BEHAVIOR**

Given an initial polygon with $n$ vertices, PCF appears to always deform the polygon into the regular $n$-gon as the figure collapses. This behavior holds regardless of the initial convexity of the polygon. One way to measure the regularity of a polygon is to look at the isoperimetric ratio (IR)

$$IR = \frac{4\pi A}{p^2},$$

where $A$ is the area of the polygon and $p$ is its perimeter. Below, the isoperimetric ratio is computed while PCF runs on a convex and non-convex octagon. We can see that over time, the IR of the each octagon approaches the IR of the regular octagon, despite the difference in their initial convexities.

**POLYGON CURVATURE FLOW**

We define the polygon curvature vector $\vec{\kappa}$ at a vertex to be

$$\vec{\kappa} = \pm \frac{\partial \theta}{\partial s} \vec{\nu}$$

where $\vec{\nu}$ is an inward pointing normal vector. To construct $\vec{\nu}$, we rotate the normalized vector $\vec{v}_2$ by the angle $\theta^*$, where $\theta^*$ is

$$\theta^* = \begin{cases} \left( \phi \left\| \vec{v}_1 \right\| \right) / \left\| \vec{v}_2 \right\| & \text{if } \left\| \vec{v}_1 \right\| \leq \left\| \vec{v}_2 \right\| \\ \left( \phi - \frac{\left\| \vec{v}_1 \right\|}{\left\| \vec{v}_2 \right\|} \right) / \left\| \vec{v}_1 \right\| & \text{if } \left\| \vec{v}_1 \right\| > \left\| \vec{v}_2 \right\| \end{cases}$$

Notice that the polygon curvature definition mimics the continuous curvature definition in its setup. This polygon curvature vector bisects the angle made by the incoming and outgoing vectors when they are of equal length. When the vectors are of different lengths, the curvature vector is weighted towards the longer one.

With this new curvature vector we can create the polygon curvature flow (PCF). This flow moves each vertex in the direction of its polygon curvature vector.

**CONJECTURES**

<table>
<thead>
<tr>
<th>CSF Property</th>
<th>PCF Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decreases Length</td>
<td>✓</td>
</tr>
<tr>
<td>Decreases Area</td>
<td>✓</td>
</tr>
<tr>
<td>Forces Convexity</td>
<td>*</td>
</tr>
<tr>
<td>Shrinks to a ’Point’</td>
<td>*</td>
</tr>
<tr>
<td>Finite Existence Time</td>
<td>✓</td>
</tr>
<tr>
<td>Maintains Simplicity</td>
<td>X</td>
</tr>
<tr>
<td>Avoidance Property</td>
<td>X</td>
</tr>
</tbody>
</table>

We have shown PCF monotonically decreases both the length and area of the polygon over time. We conjecture that PCF makes non-convex shapes convex, and then shrinks them to a regular $n$-gon ‘point.’

However, PCF does not guarantee that an initially simple polygon will stay simple, or that solutions of PCF will avoid one another, as they do with CSF.

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**DO POLYGONS BECOME ASYMPTOTICALLY REGULAR UNDER FLOW BY CURVATURE?**

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**Figure 1:** A convex octagon under PCF with its IR plotted over time (blue) vs. the regular octagon IR (red)

**Figure 2:** A non-convex octagon under PCF with its IR plotted over time (blue) vs. the regular octagon IR (red)

**Table 1:** ✓ = proven, * = conjectured, X = counterexample