Do Polygons Become Asymptotically Regular Under Flow by Curvature?

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Introduction

The process of deforming a curve by the curvature vector at each point is known as the curve-shortening flow (CSF). Grayson showed that CSF averages the shape of a curve, causing simple, closed curves to become asymptotically circular. This research applies the idea of curvature flow to polygons.

We give a novel definition for the 'curvature vector' at a vertex of a polygon. Deforming in the direction of this curvature vector yields a flow of polygons, the polygon curvature flow (PCF). We numerically investigate the behavior of this flow and show that it exhibits several of the qualitative properties of CSF. We conjecture that PCF makes polygons asymptotically regular.

Curve-Shortening Flow

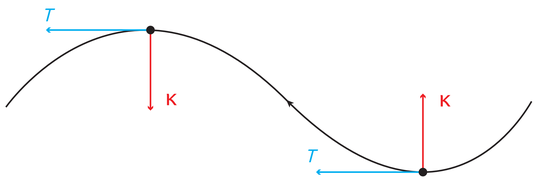
The **curve-shortening flow (CSF)** moves each point of a curve Γ in the direction of its curvature vector $\vec{\kappa}$ by applying the 'heat equation' to that curve

$$\frac{\partial \Gamma}{\partial t} = \frac{\partial^2 \Gamma}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial \Gamma}{\partial s}\right) = \frac{\partial \vec{T}}{\partial s} = \vec{\kappa}$$

One way to define the curvature vector $\vec{\kappa}$ is

$$\vec{\kappa} = \frac{d\theta}{ds}\vec{\nu}$$

where $\vec{\nu}$ is the inward pointing normal, and $\frac{d\theta}{ds}$ is the change in angle between the tangent vector and any reference vector.



Grayson's Theorem:
Given a closed, simple, smooth curve in the plane that evolves according to the curve-shortening flow, that curve will always shrink to a point in finite time, regardless of the initial convexity.

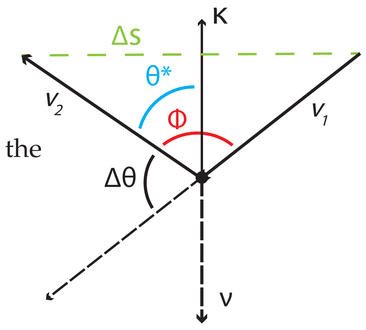
Polygon Curvature Flow

We define the polygon curvature vector $\vec{\kappa}$ at a vertex to be

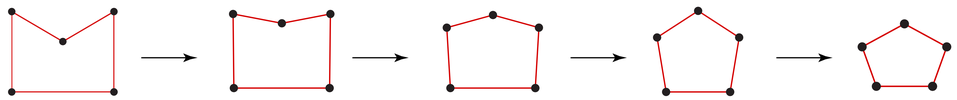
$$\vec{\kappa} = \pm \frac{\Delta\theta}{\Delta s}\vec{\nu}$$

where $\vec{\nu}$ is an inward pointing normal vector. To construct $\vec{\nu}$, we rotate the normalized vector $\vec{v_2}$ by the angle θ^*, where θ^* is

$$\theta^* = \begin{cases} \pm\left(\dfrac{\phi\|\vec{v_1}\|}{2\|\vec{v_2}\|}\right) & \text{if } \|\vec{v_1}\| \le \|\vec{v_2}\| \\ \pm\left(\phi - \dfrac{\phi\|\vec{v_2}\|}{2\|\vec{v_1}\|}\right) & \text{if } \|\vec{v_1}\| > \|\vec{v_2}\| \end{cases}$$



Notice that the polygon curvature definition mimics the continuous curvature definition in its setup. This polygon curvature vector bisects the angle made by the incoming and outgoing vectors when they are of equal length. When the vectors are of different lengths, the curvature vector is weighted towards the longer one. With this new curvature vector we can create the **polygon curvature flow (PCF)**. This flow moves each vertex in the direction of its polygon curvature vector.



Regularity and End Behavior

Given an initial polygon with n vertices, PCF appears to always deform the polygon into the regular n-gon as the figure collapses. This behavior holds regardless of the initial convexity of the polygon. One way to measure the regularity of a polygon is to look at the **isoperimetric ratio (IR)**

$$IR = \frac{4\pi A}{p^2}$$

where A is the area of the polygon and p is its perimeter. Below, the isoperimetric ratio is computed while PCF runs on a convex and non-convex octagon. We can see that over time, the IR of the each octagon approaches the IR of the regular octagon, despite the difference in their initial convexities.

Figure 1: A convex octagon under PCF with its IR plotted over time (*blue*) vs. the regular octagon IR (*red*)

Figure 2: A non-convex octagon under PCF with its IR plotted over time (*blue*) vs. the regular octagon IR (*red*)

Conjectures

CSF Property	PCF Property
Decreases Length	✓
Decreases Area	✓
Forces Convexity	✳
Shrinks to a 'Point'	✳
Finite Existence Time	✳
Maintains Simpleness	✕
Avoidance Property	✕

Table 1: ✓ = proven, ✳ = conjectured, ✕ = counterexample

We have shown PCF monotonically decreases both the length and area of the polygon over time. We conjecture that PCF makes non-convex shapes convex, and then shrinks them to a regular n-gon 'point.'

However, PCF does not guarantee that an initially simple polygon will stay simple, or that solutions of PCF will avoid one another, as they do with CSF.

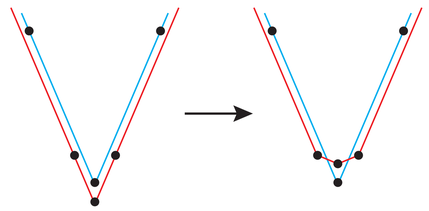


Figure 3: Counterexample of the avoidance property

The above figure demonstrates how simpleness and avoidance can be broken. Due to how Δs is defined, the outermost points are able to initially move faster and cross the innermost points. We conjecture that the flow will correct the figure back to being simple by the extinction time.

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