Geometric Proofs of Some Recent Results of Yang Lu

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Computer-aided theorem-proving has been the object of recent research in China. The pioneering work of Wu Wentsun [5] described an algebraic algorithm that was successful in proving nontrivial theorems in fields including elementary geometry. More recently, Zhang Jingzhong and Yang Lu [8] have devised an algorithm that replaces most of the symbolic algebra used in the usual methods of automated theorem-proving with numerical computation. The Zhang-Yang method truthfully can be described as a "proof by sufficient specific examples."

The object of this note is to give elementary geometric proofs of the following two results recently proved by Yang Lu via his automated method [6], [7], [9].

THEOREM 1 [6]. *In any nondegenerate tetrahedron, if the four altitudes have lengths a_1, a_2, a_3, and a_4 and the three bialtitudes have lengths b_1, b_2, and b_3, then*

$$\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}+\frac{1}{a_4^2}=\frac{1}{b_1^2}+\frac{1}{b_2^2}+\frac{1}{b_3^2}.$$

THEOREM 2 [3], [6], [7], [9]. *If the angle sum of a spherical triangle is 2π, then its medial triangle is equilaterial and has angle sum $3\pi/2$ (i.e., it is an octant).*

These two results are certainly very surprising. They are by no means widely known results and are definitely absent from the standard texts on geometry. In fact, we have been unable to find any prior reference to Theorem 1 in the literature. This is particularly curious since, as we show below, Theorem 1 is just a simple exercise in vector algebra. Theorem 2 does appear in the literature however, though the result is extremely obscure. The only reference to it that we have managed to unearth occurs in the 1909 text on spherical trigonometry by M'Clelland and Preston [3], who record it as a London University exam paper question! As we show below, Theorem 2 can also be given an elegant geometric proof.

Let us first turn to the proof of Theorem 1. Consider a tetrahedron T in $\mathbb{R}^3$. Let us suppose that it has one of its vertices at the origin and the other three at the vectors t_1, t_2, and t_3 respectively. Now recall (see for instance [1]) that there is a unique parallelepiped P in $\mathbb{R}^3$ such that the edges of the tetrahedon T are diagonals of the faces of P. Indeed, P is the parallelepiped determined by the vectors

$$p_1=(-t_1+t_2+t_3)/2,\quad p_2=(t_1-t_2+t_3)/2,\quad p_3=(t_1+t_2-t_3)/2.$$

Now the bialtitudes of T are just the distances between opposite faces of P. So if v denotes the volume of P, then in vector notation one has $v=|p_1\cdot p_2\times p_3|$ and

$$\frac{1}{b_1^2}+\frac{1}{b_2^2}+\frac{1}{b_3^2}=\frac{|p_1\times p_2|^2}{v^2}+\frac{|p_2\times p_3|^2}{v^2}+\frac{|p_1\times p_3|^2}{v^2}.$$

On the other hand, for the sum of the squares of the reciprocals of the altitudes one has

$$\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}+\frac{1}{a_4^2}=\frac{|t_1\times t_2|^2}{|t_1\cdot t_2\times t_3|^2}+\frac{|t_2\times t_3|^2}{|t_1\cdot t_2\times t_3|^2}+\frac{|t_1\times t_3|^2}{|t_1\cdot t_2\times t_3|^2}$$
$$+\frac{|(t_1-t_3)\times(t_2-t_3)|^2}{|t_1\cdot t_2\times t_3|^2}.$$

Substituting the values $t_1=p_2+p_3$, $t_2=p_1+p_3$ and $t_3=p_1+p_2$ gives the statement of Theorem 1.

We now turn to the proof of Theorem 2. The key idea is to observe that the 2-sphere is tiled by four copies of the spherical triangle $\triangle$. Indeed, by hypothesis, $\triangle$ has angle sum 2π, so at any point, o say, on the 2-sphere, one can position three copies of $\triangle$ so that the union of their regions defines a fourth triangular region, Ω say. Then the spherical triangle that bounds Ω is clearly congruent to $\triangle$. Hence S^2 is tiled by four copies of $\triangle$. Let us call these triangles $\triangle_1$, $\triangle_2$, $\triangle_3$, and $\triangle_4$.

Now, in each $\triangle_i$, draw the corresponding medial triangle Σ_i. We claim that the union of the sides of the Σ_i forms three great circles C_1, C_2, and C_3. Indeed, this follows from a straightforward congruence argument. Notice for example that, in FIGURE 1, the triangles opq and srq are congruent and hence the sides pq and qr, of Σ_1 and Σ_2 respectively, both lie on the same great circle. Identical arguments hold at each vertex of the Σ_i.

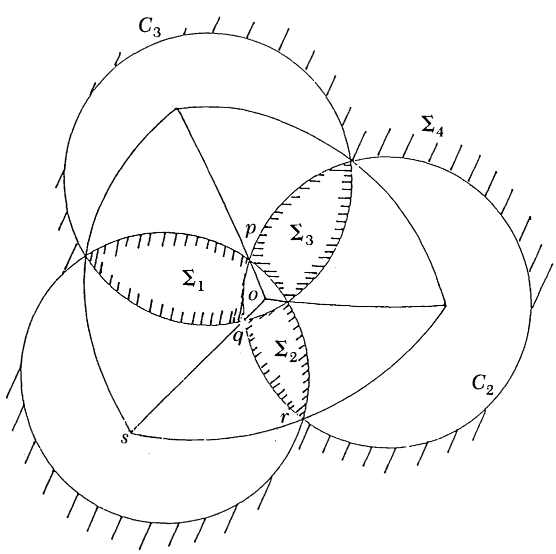

FIGURE 1

It is clear that each of the great circles C_i spends precisely one-quarter of its time in each of the triangles $\triangle_i$. Consequently the sides of the triangles Σ_i are all of length $\pi/2$ and each of the Σ_i is an octant. This completes the proof of Theorem 2.

There is a dual version of Theorem 2 (see [7]) and a converse to Theorem 2: If a spherical triangle $\triangle$ has equilateral medial triangle, then either $\triangle$ is equilateral or has angle sum 2π. We leave to the reader the pleasure of constructing geometric proofs of these results.

Finally, as commented above, Theorem 2 actually appears in a book on spherical trigonometry [3]. In fact, one can use the techniques of spherical trigonometry to obtain a much more complete picture. One has the following result.

THEOREM 3. *Let* $\triangle$ *be a spherical triangle with medial triangle* Σ. *Then the following conditions are equivalent*:

(i) $\triangle$ *has angle sum* 2π,
(ii) Σ *is an octant*,
(iii) Σ *has angle sum* $3\pi/2$,
(iv) Σ *has perimeter* $3\pi/2$,
(v) Σ *has one side of length* $\pi/2$,
(vi) Σ *has area one half the area of* $\triangle$,
(vii) *The sum of the cosines of the lengths of the sides of* $\triangle$ *equals* -1,
(viii) $\triangle$ *has a median whose length is supplementary to half the length of the side it bisects*,
(ix) *The sides of* Σ *bisect the great circle arcs from the vertices of* $\triangle$ *to the centroid of* $\triangle$.

Once again we leave the proof of this result to the reader. Let us simply say that the equivalence between the above conditions (i), (ii), (iii), (v), and (viii) has been previously observed [3] (see also [2] and [4]) and that the equivalence between the conditions (i), (ii), (iii), (iv), (v), (vii), and (viii) may all be derived easily from standard formulas such as Cagnoli's formula and Keogh's theorem. Conditions (vi) and (ix) are also not overly difficult, but do require a little extra work.

Our thanks go to Yang Lu for preprints of his papers and for many stimulating conversations.

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