A Note on the Gaussian Integral

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The elementary derivation given below for the Gaussian integral

\[ I = \int_{0}^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \]

uses integration in Cartesian coordinates and a dilation kind of change of variable. It is a better alternative to the usual method of reduction to polar coordinates that is found in texts on advanced calculus or probability and statistics.

Let \( y = x^2 \), \( dy = 2x \, dx \), then

\[
I^2 = \int_{0}^{\infty} \left( \int_{0}^{\infty} e^{-(x^2+y^2)} \, dy \right) \, dx = \int_{0}^{\infty} \left( \int_{0}^{\infty} e^{-x^2(1+s^2)} s \, dx \right) \, ds
\]

\[
= \int_{0}^{\infty} \left[ \frac{1}{-2(1 + s^2)} e^{-x^2(1+s^2)} \right]_{0}^{\infty} \, ds = \frac{1}{2} \int_{0}^{\infty} \frac{ds}{(1 + s^2)}
\]

\[ = \frac{1}{2} \arctan s \bigg|_{0}^{\infty} = \frac{\pi}{4}. \]

The idea utilized in this derivation is implicit in a similar method used for establishing the functional equation: \( \Gamma(a)\Gamma(b) = \Gamma(a+b)B(a, b) \) for the gamma and beta function, in the special case where \( a = b = \frac{1}{2} \).