Now consider an $m \times n$ matrix A with m rows $v_1, v_2, \ldots, v_m$ in F^n and n columns $w_1, w_2, \ldots, w_n$ in F^m. Suppose first that some row of A, say the kth, is extraneous. Let A' be the matrix obtained by deleting that row. Clearly, the row rank of A' is the same as that of A. But what happens to the column rank? To answer this question, first note that $v_k = \sum_{i \neq k} c_i v_i$ for suitable scalars c_i. This means that each column w_j of A lies in the subspace $W = \{(x_1, \ldots, x_m) \in F^m : x_k = \sum_{i \neq k} c_i x_i\}$. Let $w'_1, w'_2, \ldots, w'_n \in F^{m-1}$ be the columns of A', and let $T : F^m \to F^{m-1}$ be the linear map that erases the kth coordinate. Evidently $w'_j = T(w_j)$ for each j. The restriction $T|_W : W \to F^{m-1}$ is *injective*, since for $x = (x_1, \ldots, x_m) \in W$,

$$T(x) = 0 \Rightarrow (x_i = 0 \quad \forall i \neq k) \Rightarrow \left(x_k = \sum_{i \neq k} c_i x_i = 0\right) \Rightarrow x = 0.$$

Applying Fact 2, we see that a given subset of the columns of A is linearly independent if and only if the corresponding subset of the columns of A' is linearly independent. It follows that $\operatorname{colrk}(A') = \operatorname{colrk}(A)$.

So far, we have shown that deleting an extraneous row does not affect the row or column rank of A. Interchanging rows and columns in this argument, we see similarly that deleting an extraneous column does not change the row or column rank. By repeatedly deleting extraneous rows and columns, one by one, A is thereby reduced to a $p \times q$ matrix B with no extraneous rows or columns, such that B has the same row rank and column rank as A. The rows of the matrix B constitute a list of p linearly independent elements of F^q, so that $p \leq q$ by Fact 1. Similarly, $q \leq p$. Thus, $\operatorname{rowrk}(A) = \operatorname{rowrk}(B) = p = q = \operatorname{colrk}(B) = \operatorname{colrk}(A)$.

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An Elementary Proof of an Oscillation Theorem for Differential Equations

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What can we discover about the zeros of solutions of the differential equation

$$y'' + q(t)y = 0 \tag{1}$$

using graphically appealing arguments based on elementary calculus?

Equation (1) arises in many applications of mathematics. For example, calculus students often study a model of the mass-on-spring problem (see, for example, [**5**, §6.4]), in which the mass's position function $y = y(t)$ satisfies (1) with q a positive constant B^2. The general solution of (1) is then $y(t) = A\cos(Bt + \phi)$, where A and ϕ are arbitrary constants. The cosine function crosses the t-axis infinitely many times, so the model predicts that the mass will oscillate forever.

Some physical situations lead to (1) with non-constant functions q. Students of Fourier series examine a model of the motion of a vibrating drum that leads to Bessel's equation (an equation that comes up in many models in physics). Bessel's equation can be transformed into (1) with

$$q(t) = 1 + \frac{1 - 4p^2}{4t^2},$$

where p is a constant ([**2**, p. 318] and [**3**, Ch. 5]. Like the cosine function, solutions of Bessel's equation oscillate between negative and positive values (but with diminishing amplitude). Whereas in the vibrating spring problem the oscillation of the cosine function corresponds to temporal variations in the mass's location, in the drum problem the oscillation of Bessel solutions reflects *spatial* variations: at any given moment some portions of the drumhead lie above the edge of the membrane while others lie below. Based on physical considerations, then, we expect the solutions of Bessel's equation to have zeros. But do they?

In both problems, we have to decide whether the predictions of a model agree with our physical intuition. To do that we must investigate the zeros of solutions of a differential equation.

When q isn't constant, to solve (1) and determine properties of the solutions can be an elaborate affair. (See [**1**, Ch. 8] for a derivation of properties of the solutions of Bessel's equation.) Fortunately, a family of ingenious theorems (called "oscillation theorems") provide partial information about how solutions behave even when explicit formulas for the solutions aren't available. Here's the most famous member of the family; it provides a simple way to detect zeros of solutions of (1).

Sturm Comparison Theorem. *Let y_1 and y_2 be solutions of the differential equations*

$$y'' + q_1(t)y = 0 \quad \textit{and} \quad y'' + q_2(t)y = 0,$$

respectively, on an open interval I, and suppose that $q_1(t) > q_2(t)$ for all t in I. Then y_1 has at least one zero between every two zeros of y_2.

The proof of Sturm's theorem isn't difficult [**2**, Theorem 8-4, p. 314], but it does involve a hard-to-motivate trick involving the Wronskian, and it depends on the uniqueness theorem for solutions of (1), which allows us to speak of "consecutive zeros" of solutions. The uniqueness theorem is non-trivial to establish (although [**5**, Appendix A] gives a relatively elementary proof).

Here is a corollary of Sturm's result (it follows from his theorem when q_2 is a positive constant) that's strong enough to tell us that the Bessel functions have infinitely many zeros, and whose proof uses only elementary properties such as concavity. The proof does not depend on the uniqueness theorem.

Corollary. *Let Q be a positive constant, and let q be a function continuous and bounded below by Q on* $(0, \infty)$. *If f is a solution of* (1) *there, then the set of positive roots of f is unbounded.*

The function $f(t) = \sqrt{t}$ is a solution of (1) when $q(t) = \frac{1}{16t^2} > 0$, yet f has no zeros in $(0, \infty)$. Thus the assumption that q is positive isn't enough to guarantee the conclusion of the corollary; but it is enough to require that q be positive and not too small, as the following intuitive argument suggests. Suppose that q is "large enough" positive, and that f is a solution of (1). Then $f''(t) = -q(t)f(t)$. So, whenever f is large (either positive or negative), the graph of f must bend sharply enough toward the t-axis that f soon becomes small again. Now we make these intuitive ideas precise.

Lemma. *Let Q, q, and f be as in the corollary, let a be a positive number, and suppose that $f(t) > 0$ for all $t > a$. Then there exists $b > a$ such that $f'(b) < 0$.*

Loosely speaking, the lemma holds because f is positive and satisfies (1) while $q \geq Q$, so that $(f')'$ is negative and not too small; thus f' declines fairly rapidly and eventually becomes negative.

Proof of lemma. Pick $b > a$. Then, by the Mean Value Theorem and (1), there exists c in (a, b) such that

$$\begin{aligned} f'(b) &= f'(a) + f''(c)(b-a) \\ &= f'(a) - q(c)f(c)(b-a) \leq f'(a) - Qf(c)(b-a). \end{aligned} \tag{2}$$

Now suppose that, contrary to the lemma, f' is positive on (a, ∞). Then f is increasing on $[a, \infty)$, so $f(c) \geq f(a)$. It thus follows from (2) that, for b sufficiently large, $f'(b) \leq f'(a) - Qf(a)(b-a) < 0$, contradicting our assumption that f' is positive on (a, ∞). This contradiction completes the proof. ■

Proof of corollary. Let f be a solution of (1) and let a be a positive number. It suffices to show that f has at least one zero in (a, ∞).

Suppose that f has no root in (a, ∞). Since it is a solution of (1), f is differentiable and thus continuous. So either $f(t) > 0$ for all $t > a$, or $f(t) < 0$ for all $t > a$. Both cases lead to contradictions, as follows.

First suppose that $f(t) > 0$ for all $t > a$. Then the lemma guarantees a number $b > a$ such that $f'(b) < 0$. Also, (1) and the positivity of f imply that f is concave down. Thus the tangent line to the graph of f at $t = b$ lies above the graph for $t > b$. Because $f'(b) < 0$, the tangent line crosses the t-axis somewhere to the right of $t = b$; therefore, so does the graph. This contradicts the assumption that f is positive for $t > a$.

Next suppose that $f(t) < 0$ for all $t > a$. Then $-f$ is a solution of (1) that is positive for all $t > a$, which, by the previous paragraph, is impossible. This contradiction completes the proof. ■

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