Two well-known types of geometric transformations are the isometry and the similarity. A similarity with factor \( k, k > 0 \), is defined as a bijective transformation \( f \) of the plane onto the plane such that for every segment \( XY \) the distance between \( f(X) \) and \( f(Y) \) is \( k \) times the distance between \( X \) and \( Y \). An isometry is a similarity with \( k = 1 \). For isometries, we have a well-known classification. Any isometry is one of the following transformations: a translation, a rotation, a reflection or a glide reflection.

For similarities this classification is extended as follows: any similarity transformation that is not an isometry is either a dilative rotation or a dilative reflection. The former transformation preserves orientations and is called a direct transformation, whereas the latter changes orientations and is called an opposite transformation. The classification of similarities is not fully proved in most of geometry textbooks. To our knowledge, only [1] Chapter 5 and [2] give a satisfactory proof of the above classification of similarities. In this paper, an alternative proof is presented based upon Apollonius circles.

For completeness, we recall some definitions. A central dilatation or stretch with center \( C \) and factor \( k, k \neq 0 \), is a bijective transformation \( f \) of the plane onto the plane with \( C \) as a fixed point and the property \( \vec{C}f(\vec{X}) = k \cdot \vec{C}\vec{X} \) for any point \( X \). A dilative rotation is the composition of a rotation around a center \( C \) through an angle \( \alpha \) and a stretch with the same center \( C \) and a factor \( k \). A dilative reflection is the composition of a reflection in line \( \ell \) and a stretch with center \( C \), where \( C \) is on \( \ell \). The length of a segment \( XY \) is denoted by \( |XY| \).

The proofs of the classification

By definition dilative reflections and rotations have a fixed point, their center \( C \). Similarities, as defined above, with factor \( k \neq 1 \) also have a fixed point, but this should be proved. In the 1960s A.L. Steger discovered an elegant and interesting proof of the following theorem: any non-isometric similarity has a fixed point. From this theorem one derives readily using the classification of isometries that any non-isometric similarity is a dilative rotation or a dilative reflection; see [1] chapter 5.

Our proof consists of a one-step approach. It is common to state, as in [1], that a similarity is determined by its actions on three non-collinear points, but our design starts from a segment with two points. Given two segments \( A_1B_1 \) and \( A_2B_2 \) of unequal length, a single construction yields two points that are the centers of a dilative reflection and a dilative rotation respectively, both mapping \( A_1 \) to \( A_2 \) and \( B_1 \) to \( B_2 \). Consequently, when we take a third point \( C_1 \), there are only two possible images for this point.
The proof in the current paper has been found in a classical way, described by Pappus as the method of analysis. This means: assume the problem being solved and look for the defining properties of the solution [4]. So, given two segments $A_1B_1$ and $A_2B_2$ of unequal length and length ratio $1 : k$, we seek all points $Z$, which are supposed to be the center (fixed point) of a dilatation providing the necessary mapping. $Z$ must have the properties $|ZA_2| = k \cdot |ZA_1|$ and $|ZB_2| = k \cdot |ZB_1|$. Each of these two conditions defines a locus of possible points $Z$. The loci in question are well-known: they are the so-called Apollonius circles, which we discuss in the next section. The two intersection points of those circles, if they exist, will be taken in consideration as possible centers for the dilatations.

Most of the textbooks lack a satisfactory proof for the fact that any non-isometric opposite similarity is dilative reflection. See among others [3, p. 22] and [5, p. 43]. As Coxeter stated in [1, p. 67]: the direct similarities are treated but opposite similarities seem to have been neglected. Only in [2] (after Coxeter made his statement) is a sound proof found for the fact that any opposite similarity is a dilative reflection.

The Apollonius circle

Given two arbitrary points $X_1$ and $X_2$ and a constant $k > 0, k \neq 1$, the locus of points $Z$ such that $|ZX_2| = k \cdot |Z X_1|$, is a circle $\Gamma$, the so-called Apollonius circle for $X_1$ and $X_2$ with factor $k$. See Figure 1. The center of $\Gamma$ lies on the line through $X_1$ and $X_2$. Point $P$ is defined as the interior point of $X_1X_2$ such that $|PX_2| = k \cdot |PX_1|$, whereas $P'$ is defined as the exterior point of $X_1X_2$ such that $|P'X_2| = k \cdot |P'X_1|$. Hence, both $P$ and $P'$ lie on $\Gamma$ and $PP'$ is a diameter of $\Gamma$.

![Figure 1 An Apollonius circle](image)

Any point $Z$ on the Apollonius circle $\Gamma$ satisfies $|ZX_2| = k \cdot |Z X_1|$ and this equality in combination with $|PX_2| = k \cdot |PX_1|$ implies, according to a well-known angle bisector theorem, that $ZP$ is the internal bisector of $\angle X_1ZX_2$. Consequently, for any $Z$ on $\Gamma$, there is a dilative reflection with center $Z$ and factor $k$, which maps $X_1$ onto $X_2$. Conversely, if a dilative reflection with center $Z$ and factor $k$ is given that maps $X_1$ onto $X_2$, then this center $Z$ lies on $\Gamma$. The axis is always $ZP$.

Likewise, a point $Z$ lies on $\Gamma$ if and only if it is the center of dilative rotation with factor $k$ transforming $X_1$ into $X_2$.

Notice that for any $Z$ on $\Gamma$ the external bisector of $\angle X_1ZX_2$ is given by $ZP'$, again due to fact that $|P'X_2| = k \cdot |P'X_1|$. The external bisector $ZP'$ and the internal bisector $ZP$ are perpendicular. The aforementioned dilative reflection with center $Z$, axis $ZP$, and factor $k$ is identical to the dilative reflection with center $Z$, axis $ZP'$, and factor $-k$. 
Constructing transformations using Apollonius circles

Let two segments $A_1B_1$ and $A_2B_2$ be given with $k \neq 1$, where $k$ is defined as $k = \frac{|A_2B_2|}{|A_1B_1|}$. We are looking for a dilative reflection as well as a dilative rotation transforming $A_1B_1$ into $A_2B_2$. Let $\Gamma_A$ and $\Gamma_B$ denote the Apollonius circles with factor $k$ respectively for the pair $A_1, A_2$ and the pair $B_1, B_2$. The possible centers for the dilative reflection and the dilative rotation must lie on the intersection of $\Gamma_A$ and $\Gamma_B$. Unfortunately, we do not know whether these circles intersect. First of all, we shall prove that these circles must intersect.

Let $P$ and $P'$ denote the interior and exterior intersection points of $\Gamma_A$ with segment $A_1A_2$. Analogously, the interior and exterior intersection points of $\Gamma_B$ with segment $B_1B_2$ are denoted by $Q$ and $Q'$. The segments $PP'$ and $QQ'$ are diameters of the circles. In Figure 2 all these points but not the circles are drawn and two points $D$ and $D'$ are added to produce some similar triangles to be used in the proof. The points $D$ and $D'$ lie on a line parallel to $A_1B_1$ through $B_2$ on either side of $B_2$ at distance $|A_2B_2|$. Since $A_1B_1$ and $A_2B_2$ are not parallel, $A_2$ does not lie on this line. More explicitly we define $D$ and $D'$ by $B_2D = -k \cdot A_1A$ and $B_2D' = k \cdot A_1A$.

![Figure 2](image.png)

Figure 2 Why are the Apollonius circles intersecting?

Triangle $Q A_1 B_1$ is transformed by a stretch with center $Q$ and factor $-k$ into triangle $QDB_2$ from which we conclude $|QD| = k \cdot |QA_1|$. Now $Q$ is an internal point of segment $A_1D$, as is $P$ of segment $A_1A_2$. Remembering $|PA_2| = k \cdot |PA_1|$ we conclude that $DA_2$ and $PQ$ are parallel. In a similar way using a stretch through $Q'$ with factor $k$ we can prove that $D'A_2$ and $P'Q'$ are parallel.

By the construction of the points $D$ and $D'$ we have the equality $|DB_2| = |A_2B_2| = |D'B_2|$. This implies that $\angle DA_2D' = 90^\circ$. The consequence is that $PQ$ (parallel to $DA_2$) and $P'Q'$ (parallel to $D'A_2$) are perpendicular, so they surely intersect. The intersection is called $M$. Since both $\angle PMP'$ and $\angle QMQ'$ are right angles, $M$ lies on $\Gamma_A$ as well as on $\Gamma_B$.

The dilative reflection and the dilative rotation

The dilative reflection defined by factor $k$, axis $MP$ and center $M$ maps $A_1$ onto $A_2$ and the dilative reflection with factor $k$, axis $MQ$ and center $M$ maps $B_1$ onto $B_2$. Since $M$,
$P$ and $Q$ are collinear, these two transformations are identical. So, the desired dilative reflection has been found. In the next paragraph we focus on the dilative rotation.

In general, $\Gamma_A$ and $\Gamma_B$ have besides $M$ another common point, which we call $N$. A dilative rotation with center $N$ and factor $k$ transforms $A_1$ onto $A_2$. Likewise, $N$ is the center of dilative rotation with factor $k$ mapping $B_1$ onto $B_2$. These transformations are identical, if $\angle A_1 NA_2 = \angle B_1 NB_2$. We show that this is indeed the case.

In Figure 3, $\angle PMN = \angle PP'N$ and $\angle QMN = \angle QQ'N$, due to the fact that angles inscribed in the same arc of a circle are equal. Since $\angle PMN$ is identical to $\angle QMN$, we conclude $\angle PP'N = \angle QQ'N$. The angles $\angle PNP'$ and $\angle QQ'N$ are right. It follows that the right triangles $PP'N$ and $QQ'N$ are similar. The points $A_1$ and $A_2$ divide $PP'$ in the same proportions as $B_1$ and $B_2$ divide $QQ'$. Consequently, figure $PP'NA_1A_2$ is similar to figure $QQ'NB_1B_2$. This implies $\angle A_1 NA_2 = \angle B_1 NB_2$.

![Figure 3](image)

If $\Gamma_A$ and $\Gamma_B$ are tangent in $M = N$, a slightly different derivation applies. We replace in the above derivation $\angle PMN$ and $\angle QMN$ with the angle between $PP'M$ and the common tangent line in $M = N$. Similarly to above, we can derive that $\angle PP'N = \angle QQ'N$. As a result of this equality, $PP'$ or $A_1A_2$ is parallel to $QQ'$ or $B_1B_2$. The dilative rotation reduces to a central dilatation from $M = N$.

A particular case holds when $P = Q$ or $P' = Q'$. In that case $A_1B_1$ is parallel to $A_2B_2$. If $P = Q$, then $P'Q'$, $A_1B_1$ and $A_2B_2$ are parallel and we give the line through $P = Q$ perpendicular to $P'Q'$ the role of $PP'$ in the above proof. Then $M$ is again an intersection point of $\Gamma_A$ and $\Gamma_B$. The point given by $P = Q$ plays the role of $N$. The dilative rotation is a central dilatation.

The case $P' = Q'$ can be handled analogously. Notice that the situation with $P = Q$ and simultaneously $P' = Q'$ cannot happen.

The classification of the similarities

The orientation of a triple $(A, B, C)$ of non-collinear points is either clockwise or counterclockwise according as the traversal $A$ to $B$ to $C$ and back to $A$ is clockwise or not. Let two triangles $A_1B_1C_1$ and $A_2B_2C_2$ be given, such that the lengths of the sides in the latter are $k$ times the lengths in the former, $k \neq 1$. We have shown that there is a unique dilative rotation as well as a unique dilative reflection transforming $A_1B_1$
into $A_2B_2$. If the orientations of the triples $(A_1, B_1, C_1)$ and $(A_2, B_2, C_2)$ are the same, the dilative rotation also transforms $C_1$ into $C_2$. If the orientations differ, the dilative reflection does so.

As mentioned earlier, any isometry is one of the four transformations: reflection, glide reflection, rotation, or translation. Since a reflection and a rotation are special cases of a dilative reflection and a dilative rotation respectively, we conclude that any similarity is one of the following four transformations: a translation, a dilative rotation, a dilative reflection, or a glide reflection.

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Perfect Matchings, Catalan Numbers, and Pascal’s Triangle

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We wish to present a simple combinatorial proof of a determinant formula connecting the Catalan numbers and a matrix derived from Pascal’s triangle. We prove the formula by counting perfect matchings in a suitably chosen class of graphs. Although the proof relies on results and techniques from a narrow area, we still believe that it may be interesting also for readers outside this circle, since Catalan numbers are not a very common finding in (or around) the Pascal triangle. We begin with some preliminaries about benzenoid graphs.

A **benzenoid system** is a connected collection of congruent regular hexagons arranged in a plane in such a way that two hexagons are either completely disjoint or have one common edge. To each benzenoid system we can assign a **benzenoid graph**, taking the vertices of hexagons as the vertices of the graph, and the sides of hexagons as the edges of the graph. The resulting graph is simple, planar, 2-connected, bipartite and all its finite faces are hexagons.

A **perfect matching** in a graph $G$ is a collection $M$ of edges of $G$ such that every vertex of $G$ is incident with exactly one edge from $M$. The number of different perfect matchings in a graph $G$ we denote by $\Phi(G)$.

The motivation for introducing and studying benzenoid graphs came from theoretical chemistry, where they serve as the mathematical model for benzenoid hydrocarbons, a broad and important class of polycyclic carbon and hydrogen compounds in which carbon atoms are arranged in a plane pattern of rings (or cycles) of length six.