CLASSROOM CAPSULES

An Elementary Treatment of General Inner Products

Jack E. Graver (jegraver@syr.edu), Department of Mathematics, Syracuse University, Syracuse NY 13244-1150

A typical first course on linear algebra is usually restricted to vector spaces over the reals and the usual positive-definite inner product. Hence, the proof of the result

$$\dim(\mathcal{S}) + \dim(\mathcal{S}^{\perp}) = \dim(\mathcal{V})$$

is not presented in a way that is generalizable to non positive-definite inner products or to to inner products on vector spaces over other fields. In [**1**], this author made a case for proving this result in a way that does generalize to an arbitrary symmetric, nonsingular, bilinear form for vector spaces over any field. He then went on to describe just how that could be done.

The case is based on the fact that there are many useful inner products that are not positive-definite: the usual inner product for linear codes over finite fields $\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle = \sum_{i=1}^{n} x_i y_i$ and the 4-dimensional real vector space with the inner product of special relativity $\langle (t, x, y, z), (t', x', y', z') \rangle = tt' - xx' - yy' - zz'$, to name two. The dimension theorem is still valid for arbitrary inner products, but the usual proof will not work since the stronger condition $\mathcal{S} \oplus \mathcal{S}^{\perp} = \mathcal{V}$ may not hold. Unfortunately, the remedy proposed in [**1**] involves a rather extensive alteration of the usual sequencing of the material in a typical first or second linear algebra course. In this note, a much simpler adjustment is proposed. What follows is an outline of the definitions and lemmas leading to a proof of the dimension theorem in its full generality.

To start, we develop the necessary terminology including an extra definition. Given a finite-dimensional vector space $\mathcal{V}$ over an arbitrary field, a function $\langle v, w \rangle$ mapping the pairs of vectors into the scalars such that

(i) (symmetric) $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in \mathcal{V}$,
(ii) (bilinear) $\langle v, (\alpha u + \beta w) \rangle = \alpha \langle v, u \rangle + \beta \langle v, w \rangle$, for all $u, v, w \in \mathcal{V}$ and all scalars α, β,

is a *symmetric bilinear form* on $\mathcal{V}$. If in addition,

(iii) (non-singular) $\langle v, w \rangle = 0$, for all w, implies that v is the zero vector,

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the symmetric bilinear form is called an *inner product* for $\mathcal{V}$. In an elementary course, the field is the reals and the non-singular condition is replaced by

(iii′) (positive-definite) $\langle v, v\rangle > 0$, for all non-zero vectors v.

The plan here is to show how the dimension theorem can be presented in a traditional course in a way that can be generalized to an arbitrary inner product over an arbitrary field. To do this we prove as much of the dimension theorem as possible without invoking (iii) or the positive-definite condition (iii′). Many standard results, like Lemmas 1 and 2 below, use only (i) and (ii), and we have changed their statements to reflect this.

For any symmetric bilinear form, we say v is *orthogonal* to w and write $v \perp w$ whenever $\langle v, w\rangle = 0$.

Lemma 1. *Consider a vector space with a symmetric bilinear form. If v is orthogonal to $w_1, \ldots, w_k$, then v is orthogonal to every vector in the subspace spanned by $w_1, \ldots, w_k$.*

Lemma 2. *Consider a vector space $\mathcal{V}$ with a symmetric bilinear form, and let $\mathcal{S}$ be any subspace of $\mathcal{V}$. Then*

$$\mathcal{S}^{\perp} = \{v : v \perp w, \textit{ for all } w \in \mathcal{S}\} \textit{ is also a subspace of } \mathcal{V}.$$

We need just one new lemma to prove the dimension theorem.

Lemma 3. *Consider a vector space with a symmetric bilinear form and subspaces $\mathcal{S}$ and $\mathcal{T}$.*

(i) *If* $\dim(\mathcal{S}) > \dim(\mathcal{T})$, *then $\mathcal{S}$ contains a nonzero vector that is orthogonal to every vector in $\mathcal{T}$.*

(ii) *If $\mathcal{S}$ contains no nonzero vector orthogonal to every vector in $\mathcal{T}$ and $\mathcal{T}$ contains no nonzero vector orthogonal to every vector in $\mathcal{S}$, then* $\dim(\mathcal{S}) = \dim(\mathcal{T})$.

Proof. Denote the bilinear form by $\langle\,,\rangle$. Let $b_1, \ldots, b_k$ be a basis for $\mathcal{S}$ and $d_1, \ldots, d_h$ a basis for $\mathcal{T}$, where $k > h$. Consider the h-tuples $t_i = (\langle b_i, d_1\rangle, \ldots, \langle b_i, d_h\rangle)$, for $i = 1, \ldots, k$. Since $k > h$, $\{t_1, \ldots, t_k\}$ is dependent and $\sum_{i=1}^{k} \alpha_i t_i = (0, \ldots, 0)$ for some set of scalars $\alpha_1, \ldots, \alpha_k$ not all of which are zero. But then $v = \sum_{i=1}^{k} \alpha_i b_i$ is a nonzero vector of $\mathcal{S}$, and one easily sees that

$$(\langle v, d_1\rangle, \ldots, \langle v, d_h\rangle) = \sum_{i=1}^{k} \alpha_i t_i = (0, \ldots, 0).$$

Thus $v \perp d_j$, for $j = 1, \ldots, h$, and v is a nonzero vector of $\mathcal{S}$ orthogonal to every vector in $\mathcal{T}$. Part (i) is proved, and Part (ii) follows. ■

Arbitrary symmetric bilinear forms may admit troublesome vectors of two types: vectors orthogonal to every vector in the space and vectors orthogonal to themselves. A vector v that is orthogonal to every vector in the space is called a *null vector*. Clearly, the only null vector in an inner-product space is the zero vector. Also, the only *self-orthogonal vector* in a positive-definite inner product space is the zero vector. However, nonzero null vectors and nonzero self-orthogonal vectors may exist for an arbitrary symmetric bilinear form. Indeed, self-orthogonal vectors do exist in the spaces mentioned above corresponding to codes and special relativity. It is the existence of these two types of vectors that prevent one from extending the traditional proof of the dimension theorem to the general case. To give a general proof, we simply work around these troublesome vectors.

Theorem. *Consider a vector space $\mathcal{V}$ with a symmetric bilinear form and let $\mathcal{S}$ be a subspace that contains no nonzero null vector. Then*

$$\dim(\mathcal{S}) + \dim(\mathcal{S}^{\perp}) = \dim(\mathcal{V}).$$

Proof. Let $b_1, \ldots, b_k$ be a basis for $\mathcal{S}^{\perp}$, let $b_1, \ldots, b_n$ be an extension of that basis for $\mathcal{S}^{\perp}$ to a basis for $\mathcal{V}$, and let $\mathcal{T}$ be the subspace spanned by $b_{k+1}, \ldots, b_n$. Clearly, $\dim(\mathcal{T}) + \dim(\mathcal{S}^{\perp}) = \dim(\mathcal{V})$. We need only show that $\dim(\mathcal{S}) = \dim(\mathcal{T})$.

Suppose that $v \in \mathcal{T}$ is orthogonal to every vector in $\mathcal{S}$. Then $v \in \mathcal{S}^{\perp} \cap \mathcal{T}$ and, therefore, must be the zero vector. Suppose that $v \in \mathcal{S}$ is orthogonal to every vector in $\mathcal{T}$. Then $v \perp b_i$, for all i. Thus v is a null vector and, by hypothesis, must be the zero vector. The result now follows by Part (ii) of Lemma 3. ■

Corollary. *Consider a vector space $\mathcal{V}$ with an inner product. Then, for all subspaces $\mathcal{S}$ of $\mathcal{V}$:*

(i) $\dim(\mathcal{S}) + \dim(\mathcal{S}^{\perp}) = \dim(\mathcal{V})$,

(ii) $(\mathcal{S}^{\perp})^{\perp} = \mathcal{S}$.

And, if $\mathcal{S}$ contains no nonzero self-orthogonal vectors, in particular if the inner product is positive definite,

(iii) $\mathcal{S} \oplus \mathcal{S}^{\perp} = \mathcal{V}$.

Proof. Part (i) follows at once from the theorem. To prove (ii), we note that $\mathcal{S} \subseteq (\mathcal{S}^{\perp})^{\perp}$ and that, by the theorem,

$$\dim(\mathcal{S}) = \dim(\mathcal{V}) - \dim(\mathcal{S}^{\perp}) = \dim((\mathcal{S}^{\perp})^{\perp}).$$

Part (iii) follows from (i) and the observation that vectors in $\mathcal{S} \cap \mathcal{S}^{\perp}$ are self-orthogonal. ■

Summary. A typical first course on linear algebra is usually restricted to vector spaces over the real numbers and the usual positive-definite inner product. Hence, the proof that $\dim(\mathcal{S}) + \dim(\mathcal{S}^{\perp}) = \dim(\mathcal{V})$ is not presented in a way that generalizes to non-positive–definite inner products or to vector spaces over other fields. In this note we give such a proof.

Reference

1. J. E. Graver, An alternative approach to the dimension theorem for inner product spaces, *Amer. Math. Monthly* **94** (1987) 671–679. doi:10.2307/2322224