$n \leq 10{,}000$. The curve has rank three for $n = 1254$, 2605, 2774, 3502, 4199, 4669, 4895, 6286, 6671, 7230, 7766, 8005, 9015, 9430, and 9654. Noda and Wada **[8]** has a table that is an essential part of the results given in **[7]**.

Martin Gardner (**[9, 10]**) also discusses this problem and gives some related results. He offers $100 to the first person who constructs a three-by-three magic square of distinct squares.

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General Russian Roulette

GUNNAR BLOM
Department of Mathematical Statistics, University of Lund,
Box 118, S-221 00 Lund, Sweden

JAN-ERIC ENGLUND
Swedish University of Agricultural Sciences,
Box 35, S-230 53 Alnarp, Sweden

DENNIS SANDELL
Biostatistics and Data Processing, Astra Draco AB,
Box 34, S-221 00 Lund, Sweden

1. Russian roulette Russian roulette provides a standard exercise in probability. Let us quote from **[1]**, p. 32:

> Russian roulette is played with a revolver equipped with a rotatable magazine of six shots. The revolver is loaded with one shot. The first duellist, A, rotates the magazine at random, points the revolver at his head and presses the trigger. If, afterwards, he is still alive, he hands the revolver to the other duellist, B, who acts in the same way as A. The players shoot alternately in this manner, until a shot goes off. Determine the probability that A is killed.

The answer is $6/11$.

2. Generalization In this article we will consider the following generalization. There are n participants $A_1, A_2, \dots, A_n$, where $n \geq 2$. Each person has one revolver. At each trial the probability is p that a shot goes off, independently of what happens at other trials. The participants shoot in circular order

$$A_1 A_2 \dots A_n A_1 A_2 \dots A_n \dots .$$

First, A_1 uses his revolver, and either dies or survives. Thereafter, A_2 uses his weapon, and either dies or survives, and so on until one person is left; he is the winner. We want to determine the probability P_{in}, $i = 1, 2, \dots, n$, that A_i is the winner.

In order to avoid unpleasant associations in our subsequent discussions, we will now replace the revolvers with coins, which turn up heads with probability p and tails with probability $q = 1 - p$. When a player tosses his coin and obtains heads, he disappears from the list $A_1 A_2 \dots A_n A_1 A_2 \dots$. The last person remaining on the list is the winner.

3. Two players Let two people play. If A_1 obtains heads at the first toss, he disappears and A_2 is the winner. If A_1 obtains tails, the roles of the players become interchanged. These arguments lead to the relation

$$P_{12} = p \cdot 0 + q(1 - P_{12}),$$

and so we find

$$P_{12} = \frac{q}{1+q}; \quad P_{22} = \frac{1}{1+q}.$$

If $p = 1/6$ we obtain classical Russian roulette with the probabilities $5/11$ and $6/11$, respectively.

4. First recursive solution For any number of players, the P_{in}'s can be found recursively, beginning with two players, thereafter continuing with three, and so on.

(a) *Three players.*
The players toss in the order $A_1 A_2 A_3 A_1 A_2 A_3 \dots$. There are two main cases:

(i) The first toss results in heads. Player A_1 disappears. Players A_2 and A_3 remain, and for the rest of the game they take the places of A_1 and A_2, respectively, in the problem for two players.
(ii) The first toss results in tails. Players A_1, A_2, A_3 remain, and for the rest of the game they take the places of A_3, A_1 and A_2 in the original problem for three players.

Applying these considerations three times, we obtain the system of equations

$$P_{13} = p \cdot 0 + qP_{33}$$

$$P_{23} = pP_{12} + qP_{13}$$

$$P_{33} = pP_{22} + qP_{23}.$$

We already know P_{12} and P_{22}, so solve the system with respect to P_{13}, P_{23} and P_{33}. The solution is

$$P_{13} = \frac{pq}{1+q} + \frac{q^3}{1+q+q^2}$$

$$P_{23} = \frac{q}{1+q+q^2}$$

$$P_{33} = \frac{p}{1+q} + \frac{q^2}{1+q+q^2}.$$

(b) *Four players.*
When four players participate, we first solve the problem for three players and determine the P_{i4}'s from the system of equations

$$P_{14} = p \cdot 0 + qP_{44}$$

$$P_{24} = pP_{13} + qP_{14}$$

$$P_{34} = pP_{23} + qP_{24}$$

$$P_{44} = pP_{33} + qP_{34}.$$

It is now clear how the problem is solved for any given number of players:
We have $P_{1n} = qP_{nn}$ and

$$P_{in} = pP_{i-1,n-1} + qP_{i-1,n},$$

where $i = 2, \dots, n$.

5. Second recursive solution We begin the second recursive solution by constructing a recursion for P_{1n}.

If at the first toss A_1 obtains heads, he does not win the game; on the other hand, if he obtains tails, he will appear at the beginning of the second round. Suppose that there are $k+1$ people on the list after the first round. This happens if k of the players $A_2, \dots, A_n$ obtain tails during the first round; according to the binomial distribution this happens with probability

$$\binom{n-1}{k} q^k p^{n-1-k}.$$

On the other hand, when there are $k+1$ people on the list, the probability that, counted from the second round onwards, A_1 wins the games is $P_{1,k+1}$. Summing over the binomial probabilities we obtain the recursion

$$P_{1n} = q \sum_{k=0}^{n-1} \binom{n-1}{k} q^k p^{n-1-k} P_{1,k+1},$$

starting with $P_{11} = 1$.

We are now able to construct a recursion for P_{in}, $i \geq 2$. Suppose that, in the first round, k of the players $A_1, \dots, A_{i-1}$ obtain heads. This happens with probability

$$\binom{i-1}{k} p^k q^{i-1-k}.$$

When A_i tosses his coin in the first round, he is first in a game with $n-k$ people, so he wins with probability $P_{1,n-k}$. Summing over the binomial probabilities, we obtain the recursion

$$P_{in}=\sum_{k=0}^{i-1}\binom{i-1}{k}p^kq^{i-1-k}P_{1,n-k}.$$

By first computing a suitable number of P_{1j}'s, we are able to find P_{in} for any $i\geq 2$ and n.

This recursive method requires a smaller number of operations than the method described in the previous section.

6. Explicit solution We will now derive an explicit expression for the probability P_{in} that A_i wins. Let us then suppose that the game is prolonged until the winner, though being alone, goes on tossing until he obtains heads. In the main part of the solution we will assume that $1<i<n$.

Let B_j be the event that A_i obtains heads for the first time in the $(j+1)$st round, where $j=0,1,\dots$. (Remember the prolongation of the game.) The events B_j are, of course, disjoint, and we have $P(B_j)=q^jp$. If B_0 occurs, A_i can never win, so we exclude this case. Given that B_j, $j>0$, occurs, A_i wins if the following events C_j and D_j occur:

C_j: Players $A_1, A_2,\dots,A_{i-1}$ obtain heads before A_i, that is, in the $(j+1)$st round or earlier.

D_j: Players $A_{i+1}, A_{i+2},\dots,A_n$ obtain heads before A_i, that is, in the jth round or earlier.

The probability that A_1, say, obtains heads at the $(j+1)$st round or earlier is $1-q^{j+1}$. Hence we have

$$P(C_j)=(1-q^{j+1})^{i-1}.$$

Similarly we find

$$P(D_j)=(1-q^j)^{n-i}.$$

The three events B_j, C_j and D_j are independent. Summing over j we obtain

$$P_{in}=\sum_{j=1}^{\infty}P(B_jC_jD_j)=\sum_{j=1}^{\infty}P(B_j)P(C_j)P(D_j),$$

and so we arrive at the final expression

$$P_{in}=p\sum_{j=1}^{\infty}(1-q^{j+1})^{i-1}(1-q^j)^{n-i}q^j. \tag{1}$$

We leave it as an exercise to the reader to verify that this expression holds also for $i=1$. For $i=n$ the summation runs from 0 to ∞.

It follows from (1) that if $0<p<1$ then $P_{1n}<P_{2n}<\cdots<P_{nn}$. This is no surprise: Remember that A_1 begins and hence has the smallest chance to win. We also note

that $P_{1n} = qP_{nn}$; this also follows directly from the recursive relations at the end of Section 4. As a consequence, when p is small and q is therefore near 1, the P_{in}'s are almost equal.

7. Asymptotics Russian roulette with very many people involved seems unlikely. Nevertheless, friends of asymptotic solutions may like to study the behavior of (1) when n is large.

For example, when $i = 1$ it is found that

$$P_{1n} = p \sum_{j=1}^{\infty} \left(1 - q^j\right)^{n-1} q^j.$$

Replacing the sum with an integral and performing the integration we obtain

$$P_{1n} \approx -\frac{p}{n \ln q}.$$

More generally, we have

$$P_{in} \approx -\frac{p}{\ln q} \cdot \frac{1}{n - (i-1)p}.$$

The approximations become better when n grows and/or p decreases; see Table 1 for some very good values for $n = 5$.

TABLE 1. Exact and approximate winning probabilities for the two cases $n = 5$, $p = 1/6$ and $n = 5$, $p = 1/2$.

P_{i5}	$p = 1/6$		$p = 1/2$	
i	*Exact*	*Approx.*	*Exact*	*Approx.*
1	0.1828	0.1828	0.1447	0.1443
2	0.1904	0.1891	0.1628	0.1603
3	0.1989	0.1959	0.1862	0.1803
4	0.2084	0.2031	0.2169	0.2061
5	0.2194	0.2110	0.2894	0.2404

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