Buffon's Needle Problem on Radial Lines

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Duncan [1] has worked out a variation of the classical Buffon Needle Problem [2] whereby rays with uniform angular spacing $2\pi/n$ are drawn from the point O on a board, and a needle of length $2L$ and with midpoint fixed at point M is randomly thrown onto the board.

If $R = \overline{OM}$ and

$$L \leqslant R \sin \frac{\pi}{n}, \tag{1}$$

then the probability that the needle crosses a line, p, is given by

$$p = \frac{n}{\pi^2} \int_0^{\pi} \arctan\left(\frac{L \sin \phi}{R - L \cos \phi} \right) d\phi,$$

and an approximation is given by

$$p = \frac{n}{\pi^2} \log\left(\frac{R+L}{R-L} \right) + O\left(\frac{1}{n^2} \right).$$

See FIGURE 1. Note that for the case $L > R \sin(\pi/n)$, the needle may cross either or both rays forming the angle $2\pi/n$, which makes the problem much more complicated.

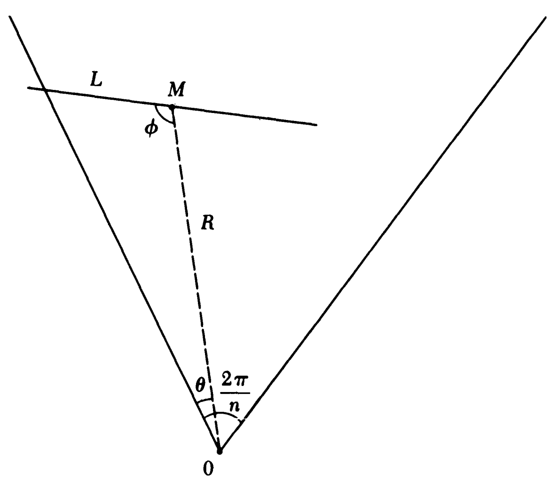

FIGURE 1
Needle of length $2L$ on set of radial lines.

This formulation of the problem involves a fixed distance, R. A problem of more interest perhaps, would be to assume that R is random such that $R \geqslant L \csc(\pi/n)$; see (1). Then, just as in Duncan's variation [1], we would let θ be the angle between OM

and the nearest line and let ϕ be the smaller angle between OM and that half of the needle that intersects the interior of the angle θ. The needle intersects one of the lines if and only if $R\sin\theta \leqslant L\sin(\theta+\phi)$ or, equivalently, if

$$\theta \leqslant \arctan\left(\frac{L\sin\phi}{R-L\cos\phi}\right).$$

We assume that θ, ϕ, and R are independent random variables, where θ is uniformly distributed on $[0,\pi/n]$ and ϕ is uniformly distributed on $[0,\pi]$.

Any realistic distribution may be attributed to the random variable R. It seems reasonable to assign a large probability to the event that the needle falls close to the point O (small R), with the probability decreasing exponentially as the needle gets farther from O (large R). The distribution describing this behavior, and one having nice limiting properties related to the classical Buffon Needle Problem, is the exponential distribution that is truncated according to (1):

$$f(R) = \frac{\lambda}{c_n}e^{-\lambda R}, \qquad R \geqslant L\csc\frac{\pi}{n},$$

with $c_n \equiv e^{-\lambda L\csc(\pi/n)}$. The parameter λ represents the reciprocal of the mean distance beyond $L\csc(\pi/n)$ attained by R.

Then, the probability that the needle intersects a radial line becomes

$$p_n = \frac{\lambda n}{c_n\pi^2}\int_{L\csc(\pi/n)}^{\infty}\left[\int_0^{\pi}\arctan\left(\frac{L\sin\phi}{R-L\cos\phi}\right)d\phi\right]e^{-\lambda R}\,dR. \tag{2}$$

For $n \geqslant 4$, we must have $L\sin\phi/(R-L\cos\phi) \leqslant 1$, so the Maclaurin series expansion for the arctangent term can be written

$$\arctan\left(\frac{L\sin\phi}{R-L\cos\phi}\right) = \frac{L\sin\phi}{R-L\cos\phi} + R_3,$$

where the remainder is

$$R_3 = \left(\frac{L\sin\phi}{R-L\cos\phi}\right)^3\left(3\delta^2\left(\frac{L\sin\phi}{R-L\cos\phi}\right)^2 - 1\right)\Bigg/3\left(1+\delta^2\left(\frac{L\sin\phi}{R-L\cos\phi}\right)^2\right)^3,$$
$$0<\delta<1.$$

Since

$$\left|3\delta^2\left(\frac{L\sin\phi}{R-L\cos\phi}\right)^2 - 1\right| \leqslant 2 \quad \text{for } n \geqslant 4,$$

we have

$$|R_3| \leqslant \left(\frac{L}{R-L}\right)^3\cdot\frac{2}{3} \leqslant \left(\frac{L}{R-L}\right)^3$$
$$\leqslant \left(\frac{\sin(\pi/n)}{1-\sin(\pi/n)}\right)^3 = O\left(\frac{1}{n^3}\right) \qquad \left(R \geqslant L\csc\frac{\pi}{n}\right).$$

Then, from equation (2),

$$p_n = \frac{\lambda n}{c_n \pi^2} \int_{L\csc(\pi/n)}^{\infty} \int_0^{\pi} \left(\frac{L \sin\phi}{R - L\cos\phi} + O\left(\frac{1}{n^3}\right) \right) e^{-\lambda R}\, d\phi\, dR$$

$$= \frac{\lambda n}{c_n \pi^2} \int_{L\csc(\pi/n)}^{\infty} \left(\log\left(\frac{R+L}{R-L}\right) + O\left(\frac{1}{n^3}\right) \right) e^{-\lambda R}\, dR.$$

And, using integration by parts (set u equal to the logarithmic term and dv equal to the exponential term), we get

$$p_n = \frac{n}{\pi^2} \log\left(\frac{1+\sin(\pi/n)}{1-\sin(\pi/n)} \right) - \frac{2\,nL}{\pi^2} e^{\lambda L \csc(\pi/n)} \int_{L\csc(\pi/n)}^{\infty} \frac{e^{-\lambda R}}{R^2 - L^2}\, dR + O(1/n^2). \tag{3}$$

Unfortunately, the answer involves a definite integral that must be approximated by a numerical quadrature routine.

A curious result is obtained upon taking the limit as n approaches infinity. By using L'Hôpital's Rule and the Fundamental Theorem of Calculus, the second term on the right side of (3) becomes zero and the first term becomes $2/\pi$. Hence,

$$\lim_{n\to\infty} p_n = 2/\pi.$$

This is just the probability that one obtains in the classical Buffon Needle Problem when the length of the needle is identical to the distance between parallel lines. This fact is easily verified by noting that $E(R) = L\csc(\pi/n) + 1/\lambda$, and $E(R) \approx nL/\pi$ as $n \to \infty$. Consequently, as n tends to infinity the set of radial lines approaches a set of parallel lines with uniform spacing that is approximated by the arc length $nL/\pi \cdot 2\pi/n = 2L$, which is just the length of the needle.

REFERENCES

1. R. L. Duncan, A variation of the Buffon Needle Problem, this MAGAZINE 40 (1967), 36–38.
2. B. V. Gnedenko, *Theory of Probability*, Chelsea, New York, 1962.