Figure 5 One can make a transparent snail ball using a jar, some corn syrup, and a moderately heavy metal cylinder.

One can easily build a simple transparent version of the device. Just partially fill a glass jar with corn syrup and then insert a heavy metal cylinder. I used a 3.5-inch diameter jar with a 1.5-inch diameter cylindrical piece of aluminum to act as the weight (FIGURE 5). One can then see the effect of friction by wrapping elastic bands around the jar and also varying the ramp surface. A more detailed study using advanced techniques from fluid dynamics is available in [**2**].

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Summary The snail ball is a device that rolls down an inclined plane, but very slowly, repeatedly coming to a stop and staying motionless for several seconds. The interior of the ball is hollow, with a smaller solid ball inside it, surrounded by a very viscous fluid. We show how to model the stop-and-start motion by analyzing the cycloidal curve that would correspond to the motion of the center of gravity as the ball rolls down an inclined plane.

Another Morsel of Honsberger

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In both of his interesting books [**5**] and [**6**], Ross Honsberger presents and proves the following property:

> In FIGURE 1, as A moves along the circular arc $\widehat{BC}$, $AB + AC$ attains its maximum when A is the midpoint M. (1)

In [**5**, Problem 9, pp. 16–17], he describes (1) as *intuitively obvious* and gives a proof. Actually, the last paragraph there can be thought of as another proof. In [**6**, pp. 21–24], he gives two more proofs. He attributes the first to I. van Yzeren and describes it as *full*

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of ingenuity, and attributes the second to K. A. Post and describes it as *most elegant*. He describes the problem as *unusually rich in interesting approaches*. In this note, we confirm this last phrase by giving three more proofs. We also examine how certain steps in our proofs and in the proofs in Honsberger's books are related to propositions in Euclid's *Elements* and to other problems in geometry.

Our proofs are simple, short, and transparent, and they prove the following statement that is slightly stronger than (1):

> In FIGURE 1, as A moves along the circular arc $\widehat{BC}$ from B to the midpoint M, $AB + AC$ increases. (2)

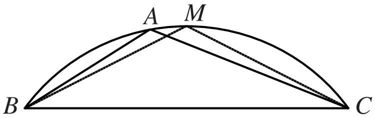

Figure 1

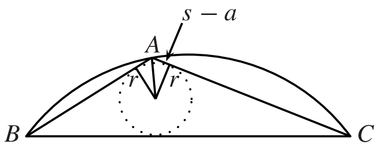

Figure 2

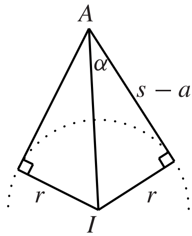

Figure 3

In these proofs, we follow common practice and we denote the side-lengths and angles of triangle ABC by a, b, c, A, B, C in the standard order, the semi-perimeter by s, the area by $[ABC]$, and the inradius by r. We also let $A = 2\alpha$, $B = 2\beta$, $C = 2\gamma$. The angle A can have any positive measure less than 180°. This means that the arc $\widehat{BC}$ can have any size, and is not necessarily a minor arc as shown in FIGURE 1.

Proof #1. Referring to FIGURE 1 and using the law of sines and simple trigonometric identities, we obtain

$$\frac{b+c}{a} = \frac{\sin 2\beta + \sin 2\gamma}{\sin 2\alpha} = \frac{2\sin(\beta+\gamma)\cos(\beta-\gamma)}{2\sin\alpha\cos\alpha} = \frac{\cos(\beta-\gamma)}{\sin\alpha}.$$

Since a and α are fixed, $b + c$ is proportional to $\cos(\beta - \gamma)$, and thus increases as A moves from B to M. ■

Proof #2. We again refer to FIGURE 1. Using the law of cosines $b^2 + c^2 - a^2 = 2bc\cos A$, the area formula $2[ABC] = bc\sin A$, and the double-angle formulas $\sin A = 2\sin\alpha\cos\alpha$, $1 + \cos A = 2\cos^2\alpha$, we obtain

$$\begin{aligned}(b+c)^2 - a^2 &= (b^2 + c^2 - a^2) + 2bc = 2bc\cos A + 2bc = 2bc(1+\cos A)\\ &= \frac{bc\sin A}{2}\,\frac{4(1+\cos A)}{\sin A} = [ABC](4\cot\alpha).\end{aligned}$$

Since a is fixed and $\cot\alpha$ is fixed and positive, it follows that $(b+c)^2$ (and hence $b + c$) increases with $[ABC]$. Since $[ABC]$ increases as A moves from B to M along $\widehat{BMC}$, so does $b + c$, as desired. ■

Proof #3. We refer to FIGURES 2 and 3. Multiplying the obvious relation $s - a = r\cot\alpha$ by 2 and then by $a + b + c$, and using $2[ABC] = r(a + b + c)$, we obtain

$$b + c - a = 2r\cot\alpha \tag{3}$$

$$(b + c)^2 - a^2 = 4[ABC]\cot\alpha. \tag{4}$$

This is the last relation in the previous proof. ∎

Beside being very short, Proof #3 has the advantage of showing that (2) still holds if $b + c$ is replaced by the inradius r. In fact, (4) implies that if $[ABC]$ increases, then $b + c$ increases, and (3) implies that if $b + c$ increases, then r increases. Of course, a is fixed and hence one can also replace $b + c$ by the perimeter $2s = a + b + c$ of ABC. We combine this with (2) in the following theorem:

THEOREM 1. *If A moves along the circular arc $\widehat{BC}$ from B to the midpoint M, then*

(i) *the area* $[ABC]$, (ii) *the perimeter* $2s$, *and* (iii) *the inradius* r

of triangle ABC increase.

So far, we have taken Part (i) of Theorem 1 for granted. Its weaker form that *the maximum of* $[ABC]$ *occurs at* $A = M$ has also been used as *obvious* in Post's proof in [**6**, p. 23]. Looking for a proof, we discovered that Theorem 1(i) appears, in disguise, within Proposition EE.III.15—meaning Book III, Proposition 15—in Euclid's *Elements*. To see this, complete the circle in FIGURE 1, draw a diameter UV parallel to BC, and drop a perpendicular AX on BC that meets UV at Y and the circle at Z; see FIGURE 4. Since a is fixed, $[ABC]$ is proportional to, and hence increases with, AX. Since XY is fixed and $AZ = 2AY$, AX increases with AZ. Thus Theorem 1(i) can be restated as follows:

In FIGURE 4, as A moves along the circular arc $\widehat{BC}$ from B to its midpoint M, AZ increases. (5)

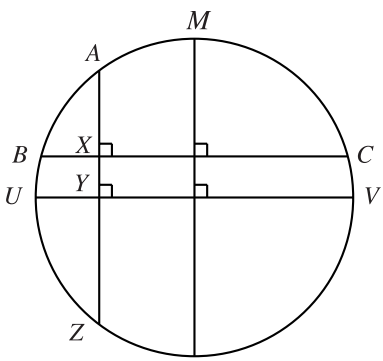

Figure 4

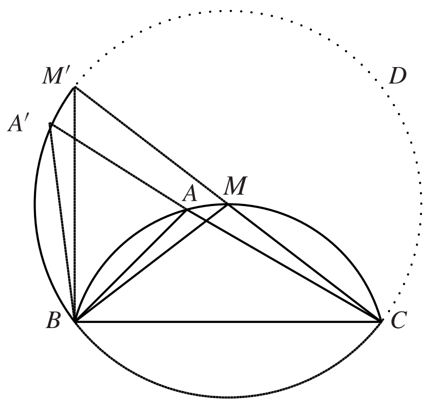

Figure 5

This statement is a special case, and the first step in the proof, of EE.III.15 which reads as follows:

PROPOSITION EE.III.15. *For two chords in a circle, the one that is nearer to the center is longer.*

Although this follows immediately from Pythagoras' theorem, Euclid's longer proof has the advantage of showing that this is a theorem in neutral geometry.

It is interesting that another obscure proposition of Book III is needed if the proof of (1) that appears in [**5**] is to be modified so that it yields (2). That proof has two components. The first consists in drawing an auxiliary circular arc centered at M and passing through B and C and lying on the same side of BC as M; see FIGURE 5. For each A on the arc $\widehat{BMC}$, one lets A' be the point where the ray CA meets the new arc. From

$$\angle AA'B + \angle ABA' = \angle BAC = \angle BMC = 2\angle BM'C = 2\angle BA'C,$$

it follows that $\angle ABA' = \angle AA'B$ and $AA' = AB$ and hence $AB + AC = AA' + AC = CA'$. In view of this, the second component of the proof (of (2)) would need the following statement:

In FIGURE 5, as a point A' moves along a semi-circle $\widehat{CBM'}$ from B to M', the length of CA' increases. (6)

But this is the special case $P = U$ of EE.III.7.

PROPOSITION EE.III.7. *If $\widehat{UV}$ is a semi-circle with center O and diameter UV and if P is between O and U but not equal to O, then as X moves from U to V on $\widehat{UV}$, the length of PX increases.*

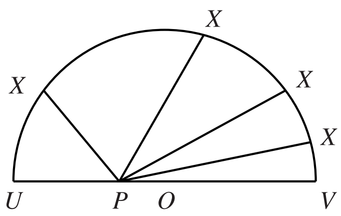

Figure 6

For the sake of completeness, we mention that Proposition EE.III.8 deals with the case when P is on the extension of OU. We also mention that it may be debatable whether Proposition EE.III.7 was meant to cover the extreme case $P = U$. This may be why this special case is added as a separate theorem in Heath's book [**4**, p. 20, last paragraph]. We also add that for proving (1) one does not need Proposition EE.III.7 but rather the simpler fact that *the length of UX attains its maximum when UX is a diameter*.

We end this note with two remarks. First, the first component of the proof in [**5**] described above is interesting on its own since it uses exactly the same configuration used in the proof of the celebrated *Broken Chord Theorem of Archimedes*; see [**7**, pp. 1–2] and compare with FIGURE 5. Secondly, it should be mentioned that our Proof #1 is inspired by a lemma that Robert Breusch had designed to solve a *Monthly* problem; see [**8**]. That lemma, together with its proof, is reproduced in [**1**] and [**2**], where it is used by the present author to give short proofs of Urquhart's theorem and of a stronger form of the Steiner-Lehmus theorem. Hyperbolic versions of Breusch's Lemma, Urquhart's Theorem, and the Steiner-Lehmus Theorem can be found in [**9**, 4.19–4.21, pp. 151–158].

(The title of this note alludes to the author's paper [**3**] on a different Honsberger topic.)

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Summary In two of his books, Ross Honsberger presented several proofs of the fact that the point A on the circular arc $\widehat{BC}$ for which $AB + AC$ is maximum is the midpoint of the arc. In this note, we give three more proofs and examine how these proofs and those of Honsberger are related to propositions in Euclid's *Elements* and, less strongly, to other problems in geometry such as the broken chord theorem, Breusch's lemma, Urquhart's theorem, and the Steiner-Lehmus theorem.

Cantor's Other Proofs that $\mathbb{R}$ Is Uncountable

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There are many *theorems* that are widely known among serious students of mathematics, but there are far fewer *proofs* that are part of our common culture. One of the best known proofs is Georg Cantor's diagonalization argument showing the uncountability of the real numbers $\mathbb{R}$. Few people know, however, that this elegant argument was not Cantor's first proof of this theorem, or, indeed, even his second! More than a decade and a half before the diagonalization argument appeared Cantor published a different proof of the uncountability of $\mathbb{R}$. The result was given, almost as an aside, in a paper [**1**] whose most prominent result was the countability of the algebraic numbers. Historian of mathematics Joseph Dauben has suggested that Cantor was deliberately downplaying the most important result of the paper in order to circumvent expected opposition from Leopold Kronecker, an important mathematician of the era who was an editor of the journal in which the result appeared [**4**, pp. 67–69]. A fascinating account of the conflict between Cantor and Kronecker can be found in Hal Hellman's book [**6**]. A decade later Cantor published a different proof [**2**] generalizing this result to perfect subsets of $\mathbb{R}^k$. This still preceded the famous diagonalization argument by six years.

Mathematical culture today is very different from what it was in Cantor's era. It is hard for us to understand how revolutionary his ideas were at the time. Many mathematicians of the day rejected the idea that infinite sets could have different cardinalities. Through much of Cantor's career many of his most important ideas were treated with skepticism by some of his contemporaries (see [**6**] for an interesting account of some of the disputes).

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