NOTES

An Advanced Calculus Approach to Finding the Fermat Point

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Steiner's problem, or Fermat's problem to Torricelli as it is sometimes called, asks for the location of the point in the plane of a given triangle whose distances from the vertices have a minimum sum. Several noncalculus solutions can be found in [**2**, pp. 24–34], [**5**, pp. 156–162], [**1**, pp. 354–361], and in many other books. However, one wonders why this beautiful extremum problem is not usually presented to students of advanced calculus. Possibly the negative speculation of D. C. Kay in his well-known *College Geometry* [**3**, p. 271] has tended to direct instructors away from the problem: "Any attempt to solve this by means of calculus would most probably end in considerable frustration." In any case, the following simple solution using calculus shows that it is well within the reach of a student of advanced calculus. It also extends to a *correct* solution of the *complementary* problem discussed by R. Courant and H. R. Robbins in their admirable *What is Mathematics*? [**1**, Chapter VII, §5.3, p. 358].

Let $P_1=(x_1,y_1)$, $P_2=(x_2,y_2)$, and $P_3=(x_3,y_3)$ be the vertices of the given triangle, and for any $P=(x,y)$ in the plane, let r_i, α_i and A_i, $i=1,2,3$, be assigned as in FIGURE 1. Thus

$$r_1=PP_1\,(=d(P,P_1)),\qquad \alpha_1=\angle P_2PP_3,\qquad A_1=\text{area of }\Delta P_2PP_3$$

and α_i is not defined if P is a vertex other than P_i. Then the function that we want to minimize is

$$f(P)=f(x,y)=r_1+r_2+r_3=\sum_{i=1}^{3}\left[(x-x_i)^2+(y-y_i)^2\right]^{1/2}$$

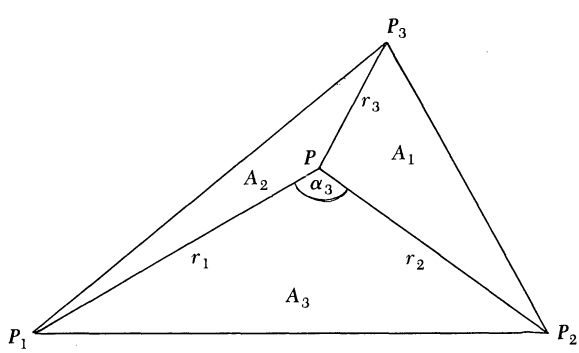

FIGURE 1

As noted in [4, p. 107], if P is properly outside the triangle then there is a line L that strictly separates P from the triangle (FIGURE 2). If Q is the foot of the line from P perpendicular to L, then for $i = 1, 2, 3$, $QP_i < PP_i$ and therefore $f(Q) < f(P)$. This shows that f has a minimum and that the minimum is attained at some point in or on the triangle.

Also, the only points in the plane at which $\partial f/\partial x$ and $\partial f/\partial y$ do not exist are the vertices of the triangle. Thus at every *critical* point $P = (x, y)$ other than these, we have

$$\partial f/\partial x = 0 \quad \text{and} \quad \partial f/\partial y = 0.$$

These equations directly yield

$$(x - x_1, x - x_2, x - x_3) \cdot (1/r_1, 1/r_2, 1/r_3) = 0$$
$$(y - y_1, y - y_2, y - y_3) \cdot (1/r_1, 1/r_2, 1/r_3) = 0$$

where "$\cdot$" represents the dot product of vectors (or points) in $\mathbb{R}^3$. Letting

$$\xi = (x - x_1, x - x_2, x - x_3), \qquad \eta = (y - y_1, y - y_2, y - y_3), \quad \text{and}$$
$$\rho = (1/r_1, 1/r_2, 1/r_3),$$

these equations are simply

$$\xi \cdot \rho = \eta \cdot \rho = 0.$$

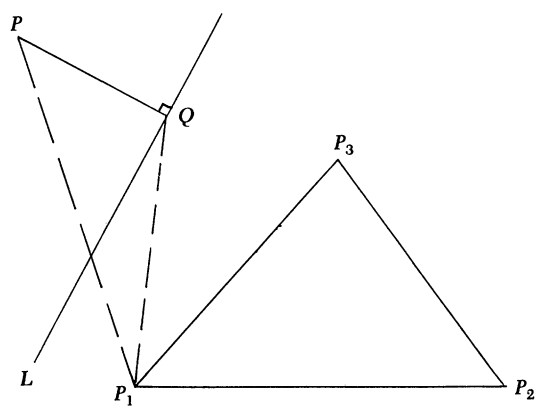

FIGURE 2

Since ξ and η are linearly independent (because P_1, P_2, and P_3 are not collinear), then ρ, being perpendicular to both ξ and η, is parallel to their cross product

$$\begin{aligned}
\xi \times \eta &= \begin{bmatrix} i & j & k \\ x - x_1 & x - x_2 & x - x_3 \\ y - y_1 & y - y_2 & y - y_3 \end{bmatrix} \\
&= \left(\begin{vmatrix} x - x_2 & x - x_3 \\ y - y_2 & y - y_3 \end{vmatrix}, \begin{vmatrix} x - x_3 & x - x_1 \\ y - y_3 & y - y_1 \end{vmatrix}, \begin{vmatrix} x - x_1 & x - x_2 \\ y - y_1 & y - y_2 \end{vmatrix} \right) \\
&= (\mp 2A_1, \mp 2A_2, \mp 2A_3) \\
&= \mp (r_2 r_3 \sin \alpha_1, r_3 r_1 \sin \alpha_2, r_1 r_2 \sin \alpha_3),
\end{aligned}$$

because all the coordinates of the parallel vector ρ have the same (positive) sign.

Therefore, for some λ,

$$\rho = (1/r_1, 1/r_2, 1/r_3) = \lambda(r_2 r_3 \sin\alpha_1, r_3 r_1 \sin\alpha_2, r_1 r_2 \sin\alpha_3).$$

Hence

$$\sin\alpha_1 = \sin\alpha_2 = \sin\alpha_3 \ (= 1/(\lambda r_1 r_2 r_3)). \tag{1}$$

Therefore, a nonvertex that minimizes f must lie inside the triangle and must have $\alpha_1 = \alpha_2 = \alpha_3 = 120°$ (for if two α_i's are supplementary for an inner point, then the third would be $180°$, contradicting (1)). If none of the angles of our triangle is $\geq 120°$, then there is a unique such point, called the (*interior*) *Fermat point* whose easy construction is given in the first figure of [**2**, p. 25] and at which f can be shown to attain its minimum. Otherwise, the minimum of f is attained at a vertex of the triangle and it is easy to see that it is attained at the vertex holding the largest angle.

It is interesting to note that if a point other than a vertex minimizes the similar expression

$$g(x, y) = r_1 + r_2 - r_3, \tag{2}$$

then the *same* equation (1) must be satisfied at that point. This is obtained by carrying out the same computations above with ρ replaced by $\tilde{\rho} = (1/r_1, 1/r_2, -1/r_3)$. However, a point P that minimizes g cannot lie inside the triangle since the intersection (FIGURE 3) Q of P_3P and P_1P_2 gives

$$g(Q) = QP_1 + QP_2 - QP_3 = P_1P_2 - QP_3 < PP_1 + PP_2 - PP_3 = g(P).$$

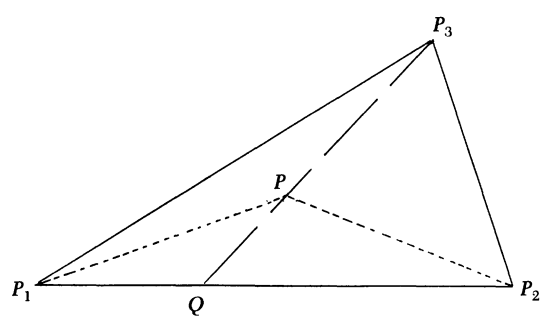

FIGURE 3

Thus a nonvertex point that minimizes g cannot lie inside the triangle and, having to satisfy (1), it must have $(\alpha_1, \alpha_2, \alpha_3) = (120°, 60°, 60°)$ in some order. Calling such points *exterior Fermat points*, we proceed to examine their existence and uniqueness and their significance in minimizing such expressions as g. We show in particular that, in contrast to the impression and statements given in [**1**], *every* triangle has an external Fermat point unless exactly one of its angles is $60°$, and that an exterior Fermat point does minimize g (or a similar expression) precisely when two angles of the triangle are $\geq 60°$ each (*and not when one of the angles is* $> 120°$).

Suppose P_0 is an exterior Fermat point of $\Delta P_1P_2P_3$. For simplicity, let α_3 be the largest among $\alpha_1, \alpha_2, \alpha_3$. Then α_3 $(= \angle P_1P_0P_2) = 120°$, $\alpha_1 = \alpha_2 = 60°$ and P_3 lies on the bisector L of $\angle P_1P_0P_2$ (FIGURE 4). Let L intersect the circle circumscribing $P_1P_0P_2$ at X and P_1P_2 at 0. Applying Ptolemy's Theorem [**3**, p. 8] to the cyclic quadrilateral $P_1P_0P_2X$ and using the fact that ΔP_1P_2X is equilateral, one concludes that

$$P_0X = P_0P_1 + P_0P_2. \tag{3}$$

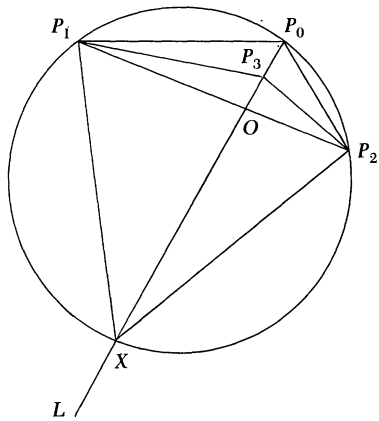

FIGURE 4

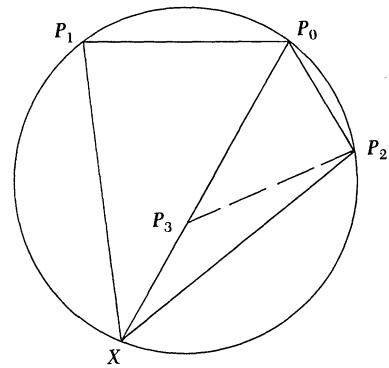

FIGURE 5

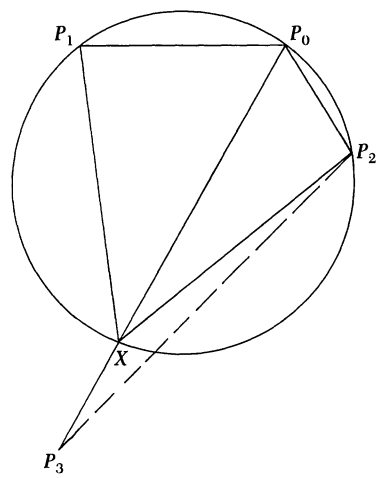

FIGURE 6

We now use this to show that if P_3 lies on the *segment* P_0X (and in particular on P_0O) (FIGURE 5) then, contrary to the claim made in [1], P_0 does *not* minimize the expression $g(P)$ of (2); while if P_3 does *not* lie on P_0X (FIGURE 6), then P_0 *does* minimize g.

Suppose that P_3 lies on the (open) segment P_0X (FIGURE 5). (Note that this includes the possibility that P_3 lies on P_0O, which corresponds to the (only) configuration considered in [1]). Then

$$g(P_0) = P_0P_1 + P_0P_2 - P_0P_3 = P_0X - P_0P_3 = P_3X$$
$$g(P_2) = P_2P_1 + 0 - P_2P_3 = P_2X - P_2P_3.$$

By the triangle inequality, we have $P_2X < P_2P_3 + P_3X$. Therefore $g(P_2) < g(P_0)$ and hence g does not take its minimum at P_0. In fact g is minimum at the vertex with the smallest angle. Also, to remove all possible remaining doubts concerning the validity of the statement about P_0 made in [1], one can similarly show that P_0 does not minimize any of the similar expressions

$$h(P) = r_2 + r_3 - r_1, \qquad k(P) = r_3 + r_1 - r_2.$$

Now suppose that P_3 does not lie on the segment P_0X (FIGURE 6). Then

$$g(P_0) = P_0P_1 + P_0P_2 - P_0P_3 = P_0X - P_0P_3 = P_3X.$$

Since

$$g(P_2) = P_2P_1 + 0 - P_2P_3 = P_2X - P_2P_3 > -P_3X = g(P_0),$$
$$g(P_1) = 0 + P_1P_2 - P_1P_3 = P_1X - P_1P_3 > -P_3X = g(P_0),$$

and since $g(P_3) > 0$, then g takes its minimum at P_0. It is also easy to see that neither h nor k is minimized at P_0.

This completes the description of the role played by an external Fermat point P_0 of a triangle $P_1P_2P_3$ in terms of how a specific vertex (say P_3) is located relative to P_0, P_1, P_2. The next theorem describes how to locate the Fermat points of a given triangle. Its first part is evident while the last part follows from examining how P_0, P_1, P_2, P_3 in FIGURES 4 and 5 are inter-located and from the previous discussion of interior Fermat points.

THEOREM. *Let $P_1P_2P_3$ be a triangle none of whose angles is $60°$ or $120°$. Let any (of the two possible) equilateral triangles, say P_1P_2X, be drawn on one of the sides. If the straight line passing through X and P_3 intersects the circle circumscribing P_1P_2X at a point F, then F is a Fermat point. Moreover, all Fermat points are obtained this way using for the side P_1P_2 the (unique) side whose two angles P_1 and P_2 are $> 60°$ each* (FIGURE 7) *or $< 60°$ each* (FIGURE 8) *and using both equilateral triangles P_1P_2X and P_1P_2X' that can be drawn on P_1P_2. Consequently, the triangle has exactly two Fermat points, one on each of the two small arcs P_1P_2.*

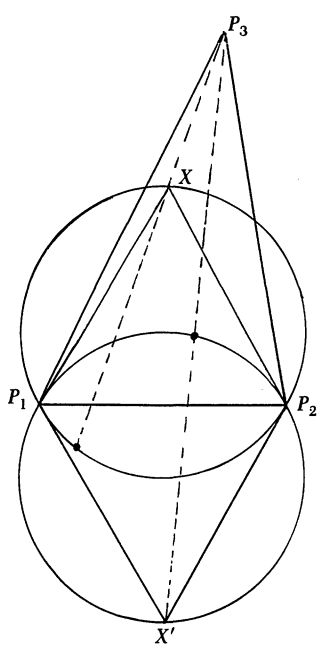

FIGURE 7

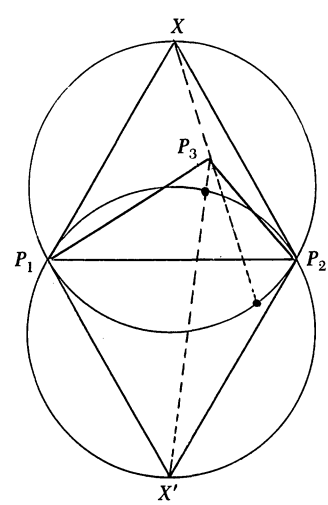

FIGURE 8

The artificial restriction on the angles of the triangle can be relaxed if we agree to call a vertex holding $60°$ or $120°$ a (*boundary*) *Fermat point*. Let us further agree to call a Fermat point *significant* if it minimizes any of the expressions f, g, h, k described above. With this terminology, the following theorem is quite evident.

THEOREM. *If a triangle is nonequilateral, then it has exactly two Fermat points. At most, one of them is boundary (and that happens precisely at the vertex, if any, at which the angle is* $60°$ *or* $120°$*) and, at most, one of them is interior (and that happens precisely when none of the angles is* $> 120°$*). Every interior or boundary Fermat point is significant while an exterior Fermat point is significant if, and only if, two angles of the triangle are* $> 60°$ *each. If the triangle is equilateral, then (by* (3)*) every point on its circumscribing circle is a significant Fermat point.*

Acknowledgment. The author thanks Ross Honsberger for his invaluable suggestions.

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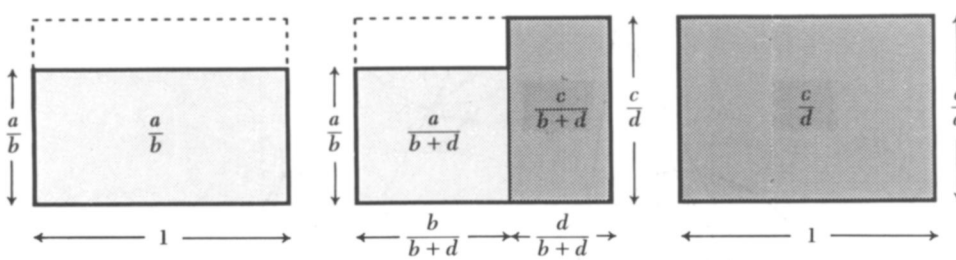
Proof without Words: Regle des Nombres Moyens [Nicolas Chuquet, *Le Triparty en la Science des Nombres*, 1484]



$$a, b, c, d > 0; \quad \frac{a}{b} < \frac{c}{d} \quad \rightarrow \quad \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

$$\frac{a}{b} < \frac{a}{b+d} + \frac{c}{b+d} < \frac{c}{d}$$

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