An Advanced Calculus Approach to Finding the Fermat Point

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Steiner’s problem, or Fermat’s problem to Torricelli as it is sometimes called, asks for the location of the point in the plane of a given triangle whose distances from the vertices have a minimum sum. Several noncalculus solutions can be found in [2, pp. 24–34], [5, pp. 156–162], [1, pp. 354–361], and in many other books. However, one wonders why this beautiful extremum problem is not usually presented to students of advanced calculus. Possibly the negative speculation of D. C. Kay in his well-known College Geometry [3, p. 271] has tended to direct instructors away from the problem: “Any attempt to solve this by means of calculus would most probably end in considerable frustration.” In any case, the following simple solution using calculus shows that it is well within the reach of a student of advanced calculus. It also extends to a correct solution of the complementary problem discussed by R. Courant and H. R. Robbins in their admirable What is Mathematics? [1, Chapter VII, §5.3, p. 358].

Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$ be the vertices of the given triangle, and for any $P = (x, y)$ in the plane, let $r_i$, $\alpha_i$ and $A_i$, $i = 1, 2, 3$, be assigned as in Figure 1. Thus

$$r_1 = PP_1 \,(= d(P, P_1)), \quad \alpha_1 = \angle P_2 PP_3, \quad A_1 = \text{area of } \Delta P_2 PP_3$$

and $\alpha_i$ is not defined if $P$ is a vertex other than $P_i$. Then the function that we want to minimize is

$$f(P) = f(x, y) = r_1 + r_2 + r_3 = \sum_{i=1}^{3} \left( (x - x_i)^2 + (y - y_i)^2 \right)^{1/2}$$

**Figure 1**
As noted in [4, p. 107], if $P$ is properly outside the triangle then there is a line $L$ that strictly separates $P$ from the triangle (Figure 2). If $Q$ is the foot of the line from $P$ perpendicular to $L$, then for $i = 1, 2, 3$, $QP_i < PP_i$ and therefore $f(Q) < f(P)$. This shows that $f$ has a minimum and that the minimum is attained at some point in or on the triangle.

Also, the only points in the plane at which $\partial f/\partial x$ and $\partial f/\partial y$ do not exist are the vertices of the triangle. Thus at every critical point $P = (x, y)$ other than these, we have

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0.$$ 

These equations directly yield

$$(x - x_1, x - x_2, x - x_3) \cdot (1/r_1, 1/r_2, 1/r_3) = 0$$

$$(y - y_1, y - y_2, y - y_3) \cdot (1/r_1, 1/r_2, 1/r_3) = 0$$

where "·" represents the dot product of vectors (or points) in $\mathbb{R}^3$. Letting

$$\xi = (x - x_1, x - x_2, x - x_3), \quad \eta = (y - y_1, y - y_2, y - y_3), \quad \rho = (1/r_1, 1/r_2, 1/r_3),$$

these equations are simply

$$\xi \cdot \rho = \eta \cdot \rho = 0.$$

Since $\xi$ and $\eta$ are linearly independent (because $P_1, P_2$, and $P_3$ are not collinear), then $\rho$, being perpendicular to both $\xi$ and $\eta$, is parallel to their cross product

$$\xi \times \eta = \begin{bmatrix} i & j & k \\ x - x_1 & x - x_2 & x - x_3 \\ y - y_1 & y - y_2 & y - y_3 \end{bmatrix} = \begin{bmatrix} |x - x_2| & |x - x_3| & |x - x_1| \\ |y - y_2| & |y - y_3| & |y - y_1| \end{bmatrix} \cdot \begin{bmatrix} i \ y - y_2 & y - y_3 \ y - y_1 \ \ y - y_3 \ y - y_1 \ y - y_2 \ end{bmatrix} = (\mp 2A_1, \mp 2A_2, \mp 2A_3) = \mp (r_2r_3 \sin \alpha_1, r_3r_1 \sin \alpha_2, r_1r_2 \sin \alpha_3),$$

because all the coordinates of the parallel vector $\rho$ have the same (positive) sign.
Thus, for some $\lambda$,

$$\rho = (1/r_1, 1/r_2, 1/r_3) = \lambda(r_2r_3 \sin \alpha_1, r_3r_1 \sin \alpha_2, r_1r_2 \sin \alpha_3).$$

Hence

$$\sin \alpha_1 = \sin \alpha_2 = \sin \alpha_3 = \frac{1}{\lambda r_1 r_2 r_3).} \tag{1}$$

Therefore, a nonvertex that minimizes $f$ must lie inside the triangle and must have $\alpha_1 = \alpha_2 = \alpha_3 = 120^\circ$ (for if two $\alpha_i$’s are supplementary for an inner point, then the third would be $180^\circ$, contradicting (1)). If none of the angles of our triangle is $\geq 120^\circ$, then there is a unique such point, called the (interior) Fermat point whose easy construction is given in the first figure of [2, p. 25] and at which $f$ can be shown to attain its minimum. Otherwise, the minimum of $f$ is attained at a vertex of the triangle and it is easy to see that it is attained at the vertex holding the largest angle.

It is interesting to note that if a point other than a vertex minimizes the similar expression

$$g(x, y) = r_1 + r_2 - r_3, \tag{2}$$

then the same equation (1) must be satisfied at that point. This is obtained by carrying out the same computations above with $\rho$ replaced by $\bar{\rho} = (1/r_1, 1/r_2, -1/r_3).$ However, a point $P$ that minimizes $g$ cannot lie inside the triangle since the intersection (Figure 3) $Q$ of $P_3P$ and $P_1P_2$ gives

$$g(Q) = QP_1 + QP_2 - QP_3 = P_1P_2 - QP_3 < PP_1 + PP_2 - PP_3 = g(P).$$

![Figure 3](image)

Thus a nonvertex point that minimizes $g$ cannot lie inside the triangle and, having to satisfy (1), it must have $(\alpha_1, \alpha_2, \alpha_3) = (120^\circ, 60^\circ, 60^\circ)$ in some order. Calling such points exterior Fermat points, we proceed to examine their existence and uniqueness and their significance in minimizing such expressions as $g$. We show in particular that, in contrast to the impression and statements given in [1], every triangle has an external Fermat point unless exactly one of its angles is $60^\circ$, and that an exterior Fermat point does minimize $g$ (or a similar expression) precisely when two angles of the triangle are $\geq 60^\circ$ each (and not when one of the angles is $> 120^\circ$).

Suppose $P_0$ is an exterior Fermat point of $\Delta P_1P_2P_3$. For simplicity, let $\alpha_3$ be the largest among $\alpha_1, \alpha_2, \alpha_3$. Then $\alpha_3 = \angle P_1P_0P_2 = 120^\circ$, $\alpha_1 = \alpha_2 = 60^\circ$ and $P_3$ lies on the bisector $L$ of $\angle P_1P_0P_2$ (Figure 4). Let $L$ intersect the circle circumscribing $P_1P_0P_2$ at $X$ and $P_1P_2$ at 0. Applying Ptolemy’s Theorem [3, p. 8] to the cyclic quadrilateral $P_1P_0P_2X$ and using the fact that $\Delta P_1P_2X$ is equilateral, one concludes that

$$P_0X = P_0P_1 + P_0P_2. \tag{3}$$
We now use this to show that if $P_3$ lies on the segment $P_0X$ (and in particular on $P_0O$) (Figure 5) then, contrary to the claim made in [1], $P_0$ does not minimize the expression $g(P)$ of (2); while if $P_3$ does not lie on $P_0X$ (Figure 6), then $P_0$ does minimize $g$.

Suppose that $P_3$ lies on the (open) segment $P_0X$ (Figure 5). (Note that this includes the possibility that $P_3$ lies on $P_0O$, which corresponds to the (only) configuration considered in [1]). Then

$$g(P_0) = P_0P_1 + P_0P_2 - P_0P_3 = P_0X - P_0P_3 = P_3X$$
$$g(P_2) = P_2P_1 + 0 - P_2P_3 = P_2X - P_2P_3.$$  

By the triangle inequality, we have $P_2X < P_2P_3 + P_3X$. Therefore $g(P_2) < g(P_0)$ and hence $g$ does not take its minimum at $P_0$. In fact $g$ is minimum at the vertex with the smallest angle. Also, to remove all possible remaining doubts concerning the validity of the statement about $P_0$ made in [1], one can similarly show that $P_0$ does not minimize any of the similar expressions

$$h(P) = r_2 + r_3 - r_1, \quad k(P) = r_3 + r_1 - r_2.$$
Now suppose that $P_3$ does not lie on the segment $P_0X$ (Figure 6). Then

$$g(P_0) = P_0P_1 + P_0P_2 - P_0P_3 = P_0X - P_0P_3 = P_3X.$$ 

Since

$$g(P_2) = P_2P_1 + 0 - P_2P_3 = P_2X - P_2P_3 > -P_3X = g(P_0),$$

and since $g(P_3) > 0$, then $g$ takes its minimum at $P_0$. It is also easy to see that neither $h$ nor $k$ is minimized at $P_0$.

This completes the description of the role played by an external Fermat point $P_0$ of a triangle $P_1P_2P_3$ in terms of how a specific vertex (say $P_3$) is located relative to $P_0, P_1, P_2$. The next theorem describes how to locate the Fermat points of a given triangle. Its first part is evident while the last part follows from examining how $P_0, P_1, P_2, P_3$ in Figures 4 and 5 are inter-located and from the previous discussion of interior Fermat points.

**Theorem.** Let $P_1P_2P_3$ be a triangle none of whose angles is $60^\circ$ or $120^\circ$. Let any (of the two possible) equilateral triangles, say $P_1P_2X$, be drawn on one of the sides. If the straight line passing through $X$ and $P_3$ intersects the circle circumscribing $P_1P_2X$ at a point $F$, then $F$ is a Fermat point. Moreover, all Fermat points are obtained this way using for the side $P_1P_2$ the (unique) side whose two angles $P_1$ and $P_2$ are $> 60^\circ$ each (Figure 7) or $< 60^\circ$ each (Figure 8) and using both equilateral triangles $P_1P_2X$ and $P_1P_2X'$ that can be drawn on $P_1P_2$. Consequently, the triangle has exactly two Fermat points, one on each of the two small arcs $P_1P_2$.

FIGURE 7

FIGURE 8

The artificial restriction on the angles of the triangle can be relaxed if we agree to call a vertex holding $60^\circ$ or $120^\circ$ a (boundary) Fermat point. Let us further agree to call a Fermat point **significant** if it minimizes any of the expressions $f, g, h, k$ described above. With this terminology, the following theorem is quite evident.
Theorem. If a triangle is nonequilateral, then it has exactly two Fermat points. At most, one of them is boundary (and that happens precisely at the vertex, if any, at which the angle is 60° or 120°) and, at most, one of them is interior (and that happens precisely when none of the angles is >120°). Every interior or boundary Fermat point is significant while an exterior Fermat point is significant if, and only if, two angles of the triangle are > 60° each. If the triangle is equilateral, then (by (3)) every point on its circumscribing circle is a significant Fermat point.

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REFERENCES

Proof without Words: Regle des Nombres Moyens
[Nicolas Chuquet, Le Triparty en la Science des Nombres, 1484]