Computing Definite Integrals using the Definition

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In teaching calculus we always give the definition of the definite integral but seldom *use* it. Stewart [**4**], for example, uses regular (uniform) partitions to define the definite integral, and then evaluates some definite integrals using right Riemann sums, justifying this with a theorem stated but unproven. Other authors ([**1**],[**2**]) use the definition given below, and then justify evaluating some integrals using regular partitions with the theorem that continuity implies Riemann integrability. Salas, Hille, and Etgen [**3**] do more than most authors and even some of what follows below, but ultimately appeal to the theorem that continuity implies integrability. We often *only* give our students *one* example (a constant function) of a definite integral being evaluated by using only the definition. This is in marked contrast to the derivative, where students often spend some time computing derivatives using the definition before moving on to the derivative rules.

What follows is a way to evaluate certain definite integrals using *only* the definition and does not appeal to auxiliary theorems.

Since texts vary on their presentation of the definite integral, I will use the following:

Definition. The Riemann integral $\int_a^b f(x)\,dx$ of a function f, which is defined on the closed interval $[a, b]$, is given by $\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\left(\sum_{i=1}^k f(x_i^*)\Delta x_i\right)$ if this limit exists, and where

- $P : a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = b$ is a partition of $[a, b]$,
- x_i^* is any point in the ith subinterval $[x_{i-1}, x_i]$,
- $\Delta x_i = x_i - x_{i-1}$ is the length of the ith subinterval, and
- $\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_k\}$ is the length of the longest subinterval.

Polynomials. Consider the function given by $f(x) = x^n$ on the interval $[a, b]$, where n is a positive integer. This method works when x is positive, so we assume $a \geq 0$. Let L_P and U_P denote the smallest and largest possible (lower and upper) Riemann sums, respectively, for a given partition P. Since f is increasing on this interval,

$$L_P = \sum_{i=1}^{k} f(x_{i-1})\,\Delta x_i = \sum_{i=1}^{k} x_{i-1}^n \Delta x_i \quad \text{and} \quad U_P = \sum_{i=1}^{k} f(x_i)\,\Delta x_i = \sum_{i=1}^{k} x_i^n \Delta x_i. \tag{1}$$

We have

$$U_P - L_P = \sum_{i=1}^{k} (x_i^n - x_{i-1}^n)\Delta x_i \le \sum_{i=1}^{k} (x_i^n - x_{i-1}^n)\|P\| = \|P\|(b^n - a^n) \tag{2}$$

so $\lim_{\|P\|\to 0} (U_P - L_P) = 0$.

Now let

$$x_i^* = \sqrt[n]{\frac{x_i^n + x_{i-1}x_i^{n-1} + x_{i-1}^2 x_i^{n-2} + \cdots + x_{i-1}^{n-1}x_i + x_{i-1}^n}{n+1}}. \tag{3}$$

Then we can observe that

$$x_{i-1} \le x_i^* \le x_i$$

by replacing x_i with x_{i-1}, and respectively, replacing x_{i-1} with x_i in (3).

Thus a specific Riemann sum for this function is given by

$$\begin{aligned} R_P &= \sum_{i=1}^{k} f(x_i^*)\Delta x_i = \sum_{i=1}^{k} (x_i^*)^n \Delta x_i \\ &= \frac{1}{n+1}\sum_{i=1}^{k} \left(x_i^n + x_{i-1}x_i^{n-1} + x_{i-1}^2 x_i^{n-2} + \cdots + x_{i-1}^{n-1}x_i + x_{i-1}^n\right)(x_i - x_{i-1}) \\ &= \frac{1}{n+1}\sum_{i=1}^{k} \left(x_i^{n+1} - x_{i-1}^{n+1}\right) = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}. \end{aligned}$$

Since $L_P \le \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} \le U_P$ and $\lim_{\|P\|\to 0}(U_P - L_P) = 0$,

$$\lim_{\|P\|\to 0}(L_P) = \lim_{\|P\|\to 0}(U_P) = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}.$$

Therefore

$$\int_a^b x^n\,dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}.$$

This all works because x_i^* is the average value of x^n on the interval $[x_{i-1}, x_i]$. Once we know that integration is linear and that $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is between a and b, we can evaluate $\int_a^b p(x)\,dx$ for any polynomial p on any interval $[a, b]$. Proofs of these properties require delicacy and will distract students from the essential ideas, so I don't advocate trying to evaluate the definite integral for all polynomials on all intervals using only the definition.

Rational functions. It should be noted that these ideas can also be used for $f(x) = \frac{1}{x^n}$ where n is a positive integer. For $0 < a < b$, let

$$x_i^* = \sqrt[n]{\frac{(n-1)(x_{i-1}x_i)^{n-1}}{x_i^{n-2} + x_{i-1}x_i^{n-3} + x_{i-1}^2 x_i^{n-4} + \cdots + x_{i-1}^{n-3}x_i + x_{i-1}^{n-2}}}.$$

It's not too hard to show $x_{i-1} \le x_i^* \le x_i$ and $\sum_{i=1}^{k} f(x_i^*)\Delta x_i = \frac{1}{(n-1)a^{n-1}} - \frac{1}{(n-1)b^{n-1}}$.

Thus it *is* possible for our beginning calculus students to compute some definite integrals using only the definition. It does require a little finesse, and work with inequalities, but these might, in the long run, be beneficial for our students.

Summary. Students in a first semester calculus course are rarely asked to compute any integrals using only the definition of the Riemann integral. This article explains how to compute some definite integrals using only the definition and no appeal to auxiliary theorems.

References

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Waiting to Turn Left?

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Driving home through rush hour traffic after a long day is frustrating, especially if you must make left turns at intersections without left-turn arrows. How does *your* state's Department of Transportation determine whether a specific intersection warrants a left-turn arrow?

The state of Pennsylvania uses a formula which measures the number of times vehicles potentially cross paths [**1**]. Specifically, traffic engineers have defined the *conflict factor* of a one-hour period of time as the product of the number of vehicles turning left during the hour and the number of vehicles continuing straight in the opposite direction. For example, if the northbound direction of traffic at a particular intersection is under consideration for a left-turn arrow, then the conflict between the southbound through traffic and the northbound left-turners is measured. The volume of eastbound and westbound traffic on the cross street is irrelevant. So, if 156 northbound cars turn left and 273 southbound cars continue straight over the course of one hour, the conflict factor is 42,588. The numbers are multiplied rather than added as this produces a better measure of conflict. If there are many cars turning left, but no opposing traffic, then there is no conflict and, therefore, no need for a dedicated arrow. (This resembles the

