$\lim_{m\to\infty}\sum_{k=0}^{m}(k/m)^m = e/(e-1)$

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In his interesting article [**4**], Michael Spivey illustrates the utility and importance of the Euler-Maclaurin formula by examining the asymptotic behavior of the power sums

$$1^m + 2^m + 3^m + \cdots + m^m$$

as $m \to \infty$. In particular, he uses this formula to evaluate the limit mentioned in the title of this Note. Here we describe two alternative ways to determine this limit. The first of these is elementary, but *ad-hoc*, while the second demonstrates the power and elegance of the Lebesgue integral, and is, perhaps, more appealing.

To set the scene, note that, if $m \geq 1$, then by reversing the sum we see that

$$\begin{aligned}\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m &= \sum_{k=0}^{m}\left(\frac{m-k}{m}\right)^m \\ &= \sum_{k=0}^{m}\left(1-\frac{k}{m}\right)^m \\ &= \sum_{k=0}^{\infty} u_m(k),\end{aligned}$$

where, for $m = 1, 2, \ldots,$

$$u_m(k) = \begin{cases}\left(1-\frac{k}{m}\right)^m, & \text{if } 0 \leq k \leq m, \\ 0, & \text{if } m \leq k.\end{cases}$$

Since the geometric-mean of a finite set of positive numbers does not exceed the arithmetic-mean of the same set of numbers, it follows that, if $1 \leq k \leq m$,

$$\sqrt[m+1]{\left(1-\frac{k}{m}\right)^m \cdot 1} \leq \frac{m(1-\frac{k}{m})+1}{m+1} = 1 - \frac{k}{m+1},$$

whence, for all $k \geq 0$,

$$0 \leq u_m(k) \leq u_{m+1}(k), \quad m = 1, 2, \ldots.$$

Also, it's familiar that

$$\lim_{m\to\infty} u_m(k) = e^{-k}, \quad k = 0, 1, 2 \ldots.$$

Thus, for each integer $k \geq 0$, the sequence $m \to u_m(k)$ increases to e^{-k}.

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Armed with these facts, let us now return to the task of evaluating the limit

$$\lim_{m\to\infty}\sum_{k=0}^{\infty} u_m(k).$$

If it were legitimate to interchange the limit operation and the summation displayed here, without qualification, we could conclude without further ado that

$$\lim_{m\to\infty}\sum_{k=0}^{\infty} u_m(k) = \sum_{k=0}^{\infty}\lim_{m\to\infty} u_m(k) = \sum_{k=0}^{\infty} e^{-k} = \frac{e}{e-1}. \tag{1}$$

However, if performed blindly, this maneuver may lead to an absurdity. As a cautionary example, if for $m = 1, 2, \dots,$

$$a_m(k) = \begin{cases} \frac{1}{m+1}, & \text{if } 0 \le k \le m, \\ 0, & \text{if } k \ge m, \end{cases}$$

then

$$1 = \lim_{m\to\infty}\sum_{k=0}^{\infty} a_m(k) \neq \sum_{k=0}^{\infty}\lim_{m\to\infty} a_m(k) = 0.$$

Therefore, the crux of the matter is the justification of the interchange employed in (1). We do this in two ways.

First we present an elementary approach. Since $0 \le u_m(k) \le e^{-k}$, we see that if $1 \le n \le m$, then

$$\sum_{k=0}^{n} u_m(k) \le \sum_{k=0}^{\infty} u_m(k) \le \sum_{k=0}^{\infty} e^{-k} = \frac{e}{e-1}, \tag{2}$$

But because $u_m(k) \le u_{m+1}(k)$ for all k, m, the sequence

$$m \to \sum_{k=0}^{\infty} u_m(k)$$

is monotonic increasing. It is also bounded above by $e/(e-1)$ as (2) shows. Hence, it is convergent, and its limit is finite. Also, $\lim_{m\to\infty} u_m(k) = e^{-k}$, and so, keeping n fixed, and making m tend to infinity in (2), we deduce that

$$\sum_{k=0}^{n} e^{-k} \le \lim_{m\to\infty}\sum_{k=0}^{\infty} u_m(k) \le \frac{e}{e-1}. \tag{3}$$

Finally, letting n tend to infinity in (3), we see that (1) holds. In other words, the limit

$$\lim_{m\to\infty}\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m$$

exists, and is equal to $e/(e-1)$.

(As the reader may care to confirm, essentially the same argument establishes the following general result: Suppose $(a_m(n))$ is a double sequence of nonnegative real numbers such that, for all natural numbers m, n, $a_m(n) \le a_{m+1}(n)$. Then

$$\lim_{m\to\infty}\sum_{n=1}^{\infty} a_m(n) = \sum_{n=1}^{\infty}\lim_{m\to\infty} a_m(n),$$

even if not all of the limits are finite. Incidentally, the same conclusion holds under the weaker hypothesis that, for all $n \geq 1$, $\lim_{m\to\infty} a_m(n)$ exists and exceeds $a_m(n)$ for all $m \geq 1$.)

Another way to justify (1) is to set the problem in the context of the Lebesgue integral [**1**], and to use one of its crowning glories, namely, Lebesgue's Monotone Convergence Theorem. This theorem states that if (X, μ) is a measure space, and (f_n) is a sequence of measurable functions such that, for all $x \in X$,

$$0 \leq f_n(x) \leq f_{n+1}(x), \quad n = 1, 2, \dots, \quad \text{and} \quad f(x) = \lim_{n\to\infty} f_n(x),$$

then f is measurable, and

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f_n \, d\mu.$$

This can be brought into play by noting that

$$\sum_{k=0}^{\infty} u_m(k) = \int_{\mathbb{N}_0} u_m \, d\nu,$$

where ν stands for the counting measure on the set of nonnegative integers $\mathbb{N}_0$, so that, if E is any subset of $\mathbb{N}_0$, whether finite or infinite, then $\nu(E)$ is the cardinal number of E. Hence, making a direct appeal to the Monotone Convergence Theorem, we deduce that

$$\begin{aligned} \lim_{m\to\infty} \sum_{k=0}^{m} \left(\frac{k}{m}\right)^m &= \lim_{m\to\infty} \int_{\mathbb{N}_0} u_m \, d\nu \\ &= \int_{\mathbb{N}_0} e^{-k} \, d\nu(k) \\ &= \sum_{k=0}^{\infty} e^{-k} \\ &= \frac{e}{e-1}. \end{aligned}$$

Similarly, it can be proved by either of the above methods that, if s is any positive real number, then

$$\lim_{m\to\infty} \sum_{k=0}^{m} \left(\frac{k}{m}\right)^{sm} = \frac{e^s}{e^s - 1}. \tag{4}$$

This raises the question: what's the story if s is complex? Answer: (4) continues to hold provided the real part of s is positive. The quickest way to see this is to use Lebesgue's Dominated Convergence Theorem, which states that if (f_n) is a sequence of measurable functions on a measure space (X, μ) such that, for all $x \in X$, $|f_n(x)| \leq F(x)$, where $\int_X F \, d\mu < \infty$, and $f(x) = \lim_{n\to\infty} f_n(x)$, then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f_n \, d\mu.$$

We leave it as an exercise for the reader to fill in the details. Alternatively, the same result can be achieved by invoking Tannery's Theorem [**2**], [**3**], which arguably is a forerunner of Lebesgue's convergence theorems.

REFERENCES

1. Robert G. Bartle, *The Elements of Integration and Lebesgue Measure*, John Wiley, New York, 1995.
2. E. T. Copson, *An Introduction to the Theory of Functions of a Complex Variable*, Oxford University Press, New York, 1935.
3. T. J. I'A. Bromwich, *An Introduction to the Theory of Infinite Series*, 2nd ed., Macmillan, London, 1926.
4. Michael Z. Spivey, The Euler-Maclaurin formula and sums of powers, *Mathematics Magazine* **79** (2006) 61–65.

Summary Two proofs are given of the limit relation $\lim_{m\to\infty}\frac{1}{m^m}\sum_{k=0}^{m}k^m=\frac{e}{e-1}$, a result due to Michael Spivey. One is elementary, and suitable for discussion in an introductory course on Analysis; the other is more sophisticated, and uses the machinery of the Lebesgue integral. Generalizations of the result are left as exercises for the reader.

Letter to the Editor

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In my paper [**1**] I prove

$$\lim_{m\to\infty}\left[\left(\frac{1}{m}\right)^m+\left(\frac{2}{m}\right)^m+\cdots+\left(\frac{m-1}{m}\right)^m\right]=\frac{1}{e-1}. \tag{1}$$

This result can be generalized. For any fixed integer k.

$$\lim_{m\to\infty}\left[\left(\frac{1}{m}\right)^m+\left(\frac{2}{m}\right)^m+\cdots+\left(\frac{m+k}{m}\right)^m\right]=\frac{e^{k+1}}{e-1}.$$

A simple way to prove this is to observe that, for $k\geq 0$,

$$\lim_{m\to\infty}\left[\left(\frac{m}{m}\right)^m+\left(\frac{m+1}{m}\right)^m+\cdots+\left(\frac{m+k}{m}\right)^m\right]=1+e+\ldots+e^k=\frac{e^{k+1}-1}{e-1}, \tag{2}$$

and then sum Equations (1) and (2). A similar argument holds for $k<0$.

There is also a mistake in my paper, first communicated to the editors by Vito Lampret. Near the end of the paper I wish to show that

$$\lim_{m\to\infty}\sum_{k=1}^{m}\left[\frac{B_k}{k!}m^{1-k}(m(m-1)\cdots(m-k+2))\right]=\sum_{k=1}^{\infty}\frac{B_k}{k!}. \tag{3}$$

First, I express the left-hand side of Equation (3) as

$$\lim_{m\to\infty}\sum_{k=1}^{m}\frac{B_k}{k!}\left[1+O\left(\frac{1}{m}\right)\right].$$

This is correct. However, the big-O notation disguises the fact that the implicit constant in $O(1/m)$ is dependent on k. Since k ranges from 1 to m over the sum, the maximum constant for the $O(1/m)$ expressions might depend on m, with the result

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