In (4), replace column 1 by column $1 + \sum_{i=1}^{n} \{x_i$ times column$(i+1)\}$ and expand along the first row. This yields

$$x_j d = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ b_1 & a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1n} \\ b_2 & a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2j} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_n & a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= d_j.$$

Since $d \neq 0$, we obtain $x_j = \dfrac{d_j}{d}$ $(j = 1, 2, \ldots, n)$.

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Pascal Triangles and Combinations Where Repetitions Are Allowed

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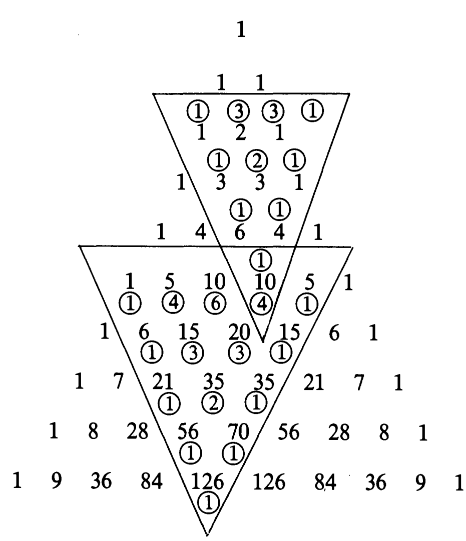
For some simple, visual representations of any entry in Pascal's triangle, superimpose an inverted Pascal triangle with the apex at the desired entry. Then the value of the entry can be found by summing the product of the entries of any row of the original triangle by the corresponding overlapping entries in the inverted triangle. This is illustrated below, where the circled numbers are the entries of the inverted Pascal triangle.

$$\begin{aligned} 10(1) &= 4(1) + 6(1) \\ &= 1(1) + 3(2) + 3(1) \end{aligned}$$

$$\begin{aligned} 126(1) &= 70(1) + 56(1) \\ &= 35(1) + 35(2) + 21(1) \\ &= 15(1) + 20(3) + 15(3) + 6(1) \\ &= 5(1) + 10(4) + 10(6) + 5(4) + 1(1) \end{aligned}$$



Since $\binom{n}{r}$ is the rth element in the nth row of Pascal's triangle (the top row is the 0th row, and the leftmost entry in each row is its 0th element), the preceding sums can be expressed as:

$$\binom{5}{3} = \sum_{i=0}^{1}\binom{4}{3-i}\binom{1}{i} = \sum_{i=0}^{2}\binom{3}{3-i}\binom{2}{i}$$

$$\binom{9}{4} = \sum_{i=0}^{1}\binom{8}{4-i}\binom{1}{i} = \sum_{i=0}^{2}\binom{7}{4-i}\binom{2}{i} = \sum_{i=0}^{3}\binom{6}{4-i}\binom{3}{i} = \sum_{i=0}^{4}\binom{5}{4-i}\binom{4}{i}.$$

This suggests the general formula

$$\binom{n+m}{r} = \binom{n}{r}\binom{m}{0} + \binom{n}{r-1}\binom{m}{1} + \cdots + \binom{n}{r-m}\binom{m}{m} \tag{1}$$

for $m \le r$ (as well as for $m > r$ if we let $\binom{n}{r-i} = 0$ when $r - i < 0$), which can be readily proved by induction on m via the identity

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}.$$

The identity (1) has an interesting application when we find the number of combinations of n objects taken r at a time where repetitions are allowed. As is well known, the number of such combinations is

$$\binom{n+r-1}{r}. \tag{2}$$

But from (1), with $m = r - 1$, we see that

$$\binom{n+r-1}{r} = \binom{n}{r} + \binom{n}{r-1}\binom{r-1}{1} + \binom{n}{r-2}\binom{r-1}{2} + \cdots + \binom{n}{1}. \tag{3}$$

Thus, we can find the number of combinations of r objects where repetitions are allowed by considering the following cases: The r objects are distinct with no repetitions, $r - 1$ objects are distinct and there is 1 repetition, $r - 2$ objects are distinct and 2 of those are repeated (those repeated may not be distinct), and so on until we select 1 object which is repeated r times.

The first factor $\binom{n}{r-i}$ gives the number of ways of choosing the $r - i$ distinct elements. The second factor, $\binom{(r-i)+i-1}{i} = \binom{r-1}{i}$, is the number of ways that we can choose i objects (which may be repeated) from the distinct $r = i$ objects. (Simply replace n by $r - i$, and r by i, in (2).)

Example. Suppose we wish to find the number of combinations of 6 objects taken 4 at a time where repetitions are allowed. Based on (3), locate the 4th element in the 6th row of Pascal's triangle and move leftward to the 1st element of the 6th row. (Remember, we begin counting with zero.) Using these values for the base of an inverted Pascal triangle, the apex will give the desired result $\binom{6+4-1}{4} = \binom{9}{4} = 126$:

1 6 15 20 15 6 1
1 3 3 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

The numbers $\binom{6}{j}$ $(j = 4, 3, 2, 1)$ in the original triangle are the number of ways we can choose j distinct objects. After selecting the objects to be used, the number of combinations using these objects is given by the respective circled numbers. Hence, the total number of combinations is

$$15(1) + 20(3) + 15(3) + 6(1) = 126.$$

This procedure also works for combinations where r is larger than n, if we extend our rows with zeros until we reach the rth entry in the nth row.

Example. The triangle below illustrates how to find the number of combinations of 3 objects taken 6 at a time, with repetitions allowed.

$$\begin{array}{ccccccccc}
 & & & 1 & 3 & 3 & 1 & 0 & 0 \quad 0 \\
 & & & & 1 & 5 & 10 & 10 & 5 \quad 1 \\
 & & & 1 & 4 & 6 & 4 & 1 & \\
 & & 1 & 5 & 10 & 10 & 5 & 1 & \\
 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & \\
1 \;\; 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 &
\end{array}$$

The nonzero elements $\binom{3}{j}$ $(j = 3, 2, 1)$ in the base of this inverted triangle can be used to find the number of combinations using 3, 2, and 1 objects. There is one way to choose 3 objects, and once they are chosen we can form 10 different combinations using those 3 objects. There are 3 different ways of choosing 2 distinct objects, and with each pair we can form 5 combinations. Finally, there are 3 different ways of choosing a single object, and with each choice we get 1 combination. Hence, the total number of combinations is

$$\binom{3+5}{6} = \binom{3}{6} + \binom{3}{5}\binom{5}{1} + \binom{3}{4}\binom{5}{2} + \binom{3}{3}\binom{5}{3} + \binom{3}{2}\binom{5}{4} + \binom{3}{3}$$

$$28 = 0 + 0 + 0 + 1(10) + 3(5) + 3(1).$$

> In the best examples, a clever person makes the computer obedient; in the worst, an obedient person hopes the computer is clever.
>
> Paul Lutus
>
> *Popular Computing* (March 1985)