We restate the above proof geometrically, using FIGURES 2 and 3. (The 3 coordinate axes are separately scaled here to bring out the salient features.) Each section "parallel" to the y-axis has the shape shown in FIGURE 1 with one dimensional critical points occurring on the ridges labeled r and on the valley bottom labeled v. Since the ridges grow like e^x the tangent plane can be horizontal only at some point of v. The only such point of v is the place where the z axis pierces v, which is the local minimum.

Proof of Theorem T

Assume f satisfies the hypotheses of the theorem. Let f have its local minimum at $A \in R^2$ and let $B \in R^2$ be such that $f(B) < f(A)$. Pour water onto the surface determined by f, above the point A. Then either (i) the water level will rise to arbitrarily great heights, or (ii) the water level will asymptotically approach a finite height h (as actually happens for f defined by (1)), or (iii) the water will overflow. We will proceed to eliminate cases (i) and (ii), in which case our earlier argument will become the proof of the theorem.

In R^2 let $\overline{AB}$ be the line segment joining A and B. Then the continuous function f attains a maximum, say m, on the compact set $\overline{AB}$. The water cannot rise to a height greater than m without spilling so that case (i) is impossible.

If case (ii) were to occur, then the water would be held by a reservoir which would lie over a subset S of the compact set $f^{-1}([f(A), h])$ and whose depth would be everywhere less than $h - f(A)$. Such a reservoir would have finite volume (less than the product of the measure of S with $h - f(A)$). Since we may pour as much water as we like, case (ii) is impossible.

References

[1] Philip Gillett, Calculus and Analytic Geometry, 2nd ed., D. C. Heath, Lexington, Mass., 1984.

"The Only Critical Point in Town" Test

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When searching for absolute extrema of functions of a single variable, it is often convenient to apply the well-known "Only Critical Point in Town" Test: *If f is a continuous function on an interval, which has a local extremum at x_0, and x_0 is the only critical point of f, then f attains an absolute extremum at x_0.* A natural question which arises is "Is the corresponding statement true for functions of two variables (say defined over the entire plane)?" Since our colleagues were evenly split on the question (both halves being quite adamant), and neither a proof nor a counterexample was readily available, we set to work trying to find one.

Progress came slowly at first. Then one bright Monday morning we exchanged pictures of what we thought the level curves of a counterexample might look like. And believe it or not, we both had the same picture!—right down to the location of the mountain, the river bed, and the cliff! It looked something like FIGURE 1. Most of our colleagues were convinced by our picture, but a few remained rightfully skeptical: They wanted a formula. We did too.

Then we noticed something—our level curve picture seemed to have a "saddle point at infinity." So we tried to mimic this.

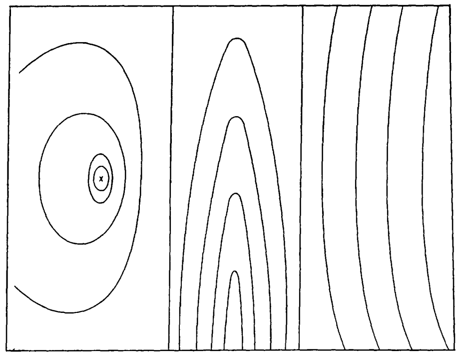

FIGURE 1

Starting with the surface

$$g(x, y) = 3xy - x^3 - y^3,$$

which has a local maximum at (1,1) and a saddle point at (0,0), to "push the saddle point to infinity," we simply took the function

$$f(x, y) = g(x, e^y) = 3xe^y - x^3 - e^{3y}.$$

With this function it is easily seen that the point (1,0) is the only critical point in the plane and that f attains a local maximum there. But it is clearly not an absolute maximum since $f(x,0) \to \infty$ as $x \to -\infty$.

In FIGURE 2 you may look at the "entire" graph of f which we have "scrunched" for easy viewing.

We would like to thank Hans Reddinger and some of his University of Wyoming "toys" for the great graphics.

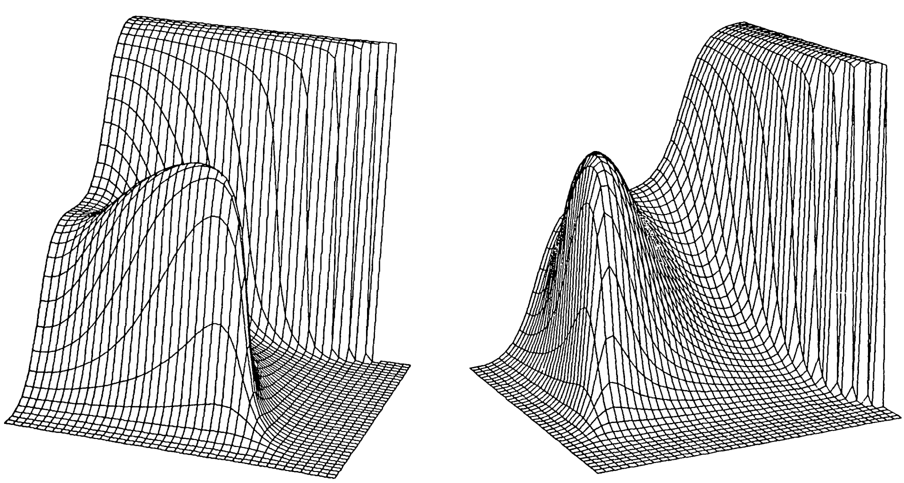

FIGURE 2. **Views of the "scrunched" version of $z = 3xe^y - x^3 - e^{3y}$, that is, $z = \arctan[3e^{\tan y} \tan x - \tan^3 x - e^{3\tan y}]$ for $-\pi/2 < x,\ y < \pi/2$.**