imaginary parts of $e^{\alpha x}f(x)/e^{rx}$ become indeterminate forms to which one can apply L'Hôpital's rule. Proceeding in a manner similar to our previous proofs, we are able to show that

$$f'(x)+\alpha f(x)\to\beta \text{ as } x\to+\infty \text{ implies } f(x)\to\beta/\alpha \text{ and } f'(x)\to 0$$

even when $\text{Re}(\alpha)<0$ if we also know that $f(x)$ is bounded for large x. An illustration of this case is given by the function $f(x)=ix/(x-i)$ and $\alpha=-1-i$; one may check that the limit β in (2) is $\beta=1-i$, and $\lim_{x\to+\infty}f(x)=\beta/\alpha=i$. Note the complete agreement of these results with those previously derived in the real case.

(iii) Last, consider $\text{Re}(\alpha)=0$. Since $\alpha\neq 0$, then $\text{Im}(\alpha)\neq 0$. In this case even requiring that f is eventually bounded and that the limit β in (2) exists does not ensure that $\lim_{x\to+\infty}f(x)$ exists. Consider the example $f(x)=e^{ix}$ and $\alpha=-i$; then $f'(x)+\alpha f(x)\equiv 0$, and the functions f, f' are bounded, but neither has a limit as $x\to+\infty$. Suppose we assume that *either* f or f' has a finite limit as $x\to+\infty$, and that $f'(x)+\alpha f(x)\to 0$ as $x\to+\infty$. Then both f and f' must have finite limits as $x\to\infty$, say γ, δ respectively, and $\delta+\alpha\gamma=0$. Then $\gamma=0$ if and only if $\delta=0$. However, if $\delta\neq 0$ then either $\text{Re}(f')$ or $\text{Im}(f')$ would have a nonzero limit which would force $|f(x)|\to+\infty$, contradicting $f(x)\to\gamma$. Therefore, the assumption that either f or f' has a finite limit leads to the conclusion that *both* of these limits are zero. More generally, with this assumption, we again may say

$$\text{if } f'(x)+\alpha f(x)\to\beta \text{ as } x\to+\infty, \text{ then } f(x)\to\beta/\alpha \text{ and } f'(x)\to 0.$$

An example of this last case is provided by the function $f(x)=1+i/x$ and $\alpha=i$; then $\beta=i$ and $\beta/\alpha=1$.

Reference [5] discusses conclusions one may draw about $\lim_{x\to+\infty}f(x)$ when certain linear combinations of f, f' and higher derivatives have finite limits. The reader might find it interesting to examine the relationship between $\lim_{x\to+\infty}[xf'(x)+rf(x)]$ and the limits of $f(x)$ and $f'(x)$. Replacing $e^xf(x)/e^x$ by $x^rf(x)/x^r$ allows one to use the methods of this paper to achieve similar results.

References

[1] W. Fulks, Advanced Calculus, Wiley, New York, 1961, p. 134.
[2] G. H. Hardy, A Course of Pure Mathematics, 1st ed., Cambridge, 1908, p. 246.
[3] ______, A Course of Pure Mathematics, 10th ed., Cambridge, 1955, p. 281.
[4] H. Kestelman, An old exercise, Amer. Math. Monthly, 85 (1978) 685–686.
[5] W. C. Waterhouse, Limit of solutions of a linear differential equation, Amer. Math. Monthly, 81 (1974) 92–93.

An Alternative to the Integral Test

GERALD JUNGCK
Bradley University
Peoria, IL 61625

As presented in Calculus texts, the Integral Test states that *if f is continuous, positive valued, and decreasing on* $[1,\infty)$, *then*

$$\sum_{n=1}^{\infty} f(n) \text{ converges iff } \int_1^{\infty} f(x)\,dx \text{ converges}.$$

Since $f(x)>0$ on $[1,\infty)$, $\int_1^{\infty}f(x)\,dx$ converges iff any antiderivative of f is bounded above. So the Integral Test is virtually an "Antiderivative Test." However, the Integral Test as stated fails to utilize the fact that we can be selective in our choice of antiderivatives.

In this note we propose as an alternative to the Integral Test an antiderivative test which can be proved and applied outside the context of the definite and improper integrals. This approach eliminates dependency on improper integrals, and also makes more feasible the option of introducing infinite series before the definite integral. Moreover, by choosing antiderivatives judiciously in this test, we obtain inequalities which prove useful in approximation and error analysis.

Our approach is based on the observation that an infinite series $\Sigma f(n)$ and its sequence $\{S_n\}$ of partial sums satisfies

$$S_{n+1}-S_n=f(n+1),$$

so it appears natural to investigate functions g such that $g(x+1)-g(x)$ approximates $f(x+1)$. Now if such a g is differentiable on $[x, x+1]$, then the Mean Value Theorem (MVT) implies that $g(x+1)-g(x)=g'(c)$ for some c between x and $x+1$; i.e., we want g' to approximate f. We are thus led to consider antiderivatives of f.

The **antiderivative test**, which we now state and prove, follows easily from the MVT and the fact that a series of positive terms converges if and only if its sequence of partial sums is bounded above.

THEOREM. *Let f be positive and nonincreasing on $[k,\infty)$, k a positive integer, and let g be any antiderivative of f. Then*

(A) *$\Sigma_{n=k}^{\infty} f(n)$ converges iff g is bounded above on $[k,\infty)$.*
(B) *If $\lim_{x\to\infty} g(x)=0$, then $\Sigma_{n=k}^{\infty} f(n)$ converges.*

Furthermore, if S is the sum of $\Sigma_{n=k}^{\infty} f(n)$, then

$$-g(m+1)\leqslant \sum_{n=m+1}^{\infty} f(n)\leqslant -g(m) \text{ for } m\geqslant k, \tag{1}$$

$$f(k)-g(k+1)\leqslant S\leqslant f(k)-g(k), \tag{2}$$

and

$$0\leqslant S-\left(\sum_{n=k}^{m} f(n)-g(m+1)\right)\leqslant f(m) \text{ for } m\geqslant k. \tag{3}$$

Proof of (A). For $n\geqslant k$ we can apply the MVT to g on $[n, n+1]$ to obtain

$$g(n+1)-g(n)=g'(x_n)((n+1)-n)=f(x_n)$$

for some $x_n\in(n, n+1)$. Since f is nonincreasing we have

$$f(n+1)\leqslant f(x_n)=g(n+1)-g(n)\leqslant f(n) \text{ for } n\geqslant k. \tag{4}$$

But $\Sigma_{n=k}^{m}(g(n+1)-g(n))=g(m+1)-g(k)$ for $m\geqslant k$, so (4) implies

$$\sum_{n=k}^{m} f(n+1)\leqslant g(m+1)-g(k)\leqslant \sum_{n=k}^{m} f(n) \text{ for } m\geqslant k. \tag{5}$$

Now if $\Sigma_{n=k}^{\infty} f(n)$ converges, the right hand term in (5) is bounded above so that (5) implies that g is bounded on the set of integers $\geqslant k$. But $g'(x)=f(x)>0$ and consequently g is increasing on $[k,\infty)$; therefore g is bounded on $[k,\infty)$.

Conversely, if g is bounded on $[k,\infty)$, (5) implies that the sequence of partial sums for the series $\Sigma_{n=k}^{\infty} f(n+1)$ is bounded above; thus $\Sigma_{n=k}^{\infty} f(n+1)$ and $\Sigma_{n=k}^{\infty} f(n)$ are convergent.

A few comments before we prove part (B) of the theorem. Observe that the requirement that $\lim_{x\to\infty} g(x)=0$ can always be met if f has an antiderivative which is bounded above. For example, if $f(n)=1/(1+n^2)$ we would let $g(x)=\tan^{-1}x-\pi/2$ instead of $\tan^{-1}x$. We introduce the following notation for the proof and subsequent examples:

$$S_m = \sum_{n=k}^{m} f(n), E_m = \sum_{n=m+1}^{\infty} f(n).$$

(Thus E_m is the error if we approximate S by S_m).

Proof of (B). If $\lim_{x\to\infty} g(x) = 0$, then g is certainly bounded on $[k,\infty)$ so that the indicated series is convergent by part (A). Now if we let $m \to \infty$ in (5), we obtain

$$\sum_{n=k}^{\infty} f(n+1) \leqslant -g(k) \leqslant \sum_{n=k}^{\infty} f(n).$$

Since (5) is valid when any $k' \geqslant k$ is substituted for k, we can write

$$\sum_{n=m}^{\infty} f(n+1) \leqslant -g(m) \leqslant \sum_{n=m}^{\infty} f(n) \text{ for } m \geqslant k. \tag{6}$$

The left side of (6) implies that $E_m \leqslant -g(m)$, and the right side of (6) implies $-g(m+1) \leqslant E_m$; consequently, inequality (1) of (B) holds (for $m \geqslant k$). Inequality (2) follows from (1); just let $m = k$ in (1), and add $f(k)$ throughout (note that $S = E_k + f(k)$). Moreover, (1) implies that

$$0 \leqslant E_m + g(m+1) = (S - S_m) + g(m+1) \leqslant g(m+1) - g(m)$$

for $m \geqslant k$, and hence (4) yields

$$0 \leqslant S - \big(S_m - g(m+1)\big) \leqslant f(m) \text{ for } m \geqslant k$$

which is inequality (3).

We now illustrate the use of the Theorem with several examples.

EXAMPLE 1. We consider the series

$$\sum_{n=1}^{\infty} n^{-2}, \tag{7}$$

and let $f(x) = x^{-2}$. If $g(x) = -x^{-1}$, then $g' = f$ and $g(x) \to 0$, so part (B) of the Theorem applies to g. By (2),

$$f(1) - g(2) = 3/2 \leqslant S \leqslant 2 = f(1) - g(1),$$

which yields an initial approximation to S. Also, (1) implies that

$$-g(m+1) = (m+1)^{-1} \leqslant E_m \leqslant m^{-1} = -g(m)$$

is true for $m \geqslant 1$. Thus for S_m to approximate S accurately to three decimal places, it suffices that $m^{-1} < .5(10^{-3})$ or $m > 2{,}000$. In fact, the above inequality shows that $m \geqslant 2{,}000$ is a necessary condition for 3 place accuracy. However, we can approximate S more economically by using (3), since

$$0 \leqslant S - \big(S_m - g(m+1)\big) \leqslant m^{-2} < 1/2{,}000 \text{ when } m > \sqrt{20}\,10.$$

Thus $S_{45} - g(46) = 1.622957 + .021739 = 1.644696$ approximates S accurately to 3 decimal places.

EXAMPLE 2. $\sum_{n=1}^{\infty} n^{-p}$ converges if $p > 1$ and diverges if $p < 1$, since if $g(x) = x^{1-p}/(1-p)$ for $x \geqslant 1$, then $g(x) \to 0$ as $x \to +\infty$ if $p > 1$, and $g(x) \to +\infty$ if $p < 1$. So when $p > 1$,

(1) implies $\quad (m+1)^{1-p}/(p-1) \leqslant E_m \leqslant m^{1-p}/(p-1)$ for $m \geqslant 1$;

(2) implies $\quad 1 + 2^{1-p}/(p-1) \leqslant S \leqslant 1 + 1/(p-1)$,

and

(3) implies that for $m \geqslant 1$, $\quad 0 \leqslant S - \big(S_m - g(m+1)\big) \leqslant m^{-p}$.

Consequently, if we use S_m to approximate S, we obtain n place accuracy if $m >$

$(2(10^n)/(p-1))^{1/(p-1)}$, and if we use $S_m - g(m+1)$ to approximate S, we obtain n place accuracy if $m > (2(10)^n)^{1/p}$.

The case $p = 1$ ignored in Example 2 can be dealt with even though the natural logorithm has not been introduced (see, e.g., [**1**], p. 566).

EXAMPLE 3. Given $\Sigma_{n=3}^{\infty} n(n^2 - 7)^{-5/2}$, let $g(x) = -(x^2 - 7)^{-3/2}/3$. Then if $x > \sqrt{83.3}$, $-g(x) < .5(10)^{-3}$, so that S_{10} approximates S accurately to three decimal places by (1).

In the above example a nice antiderivative g was readily apparent. If this is not the case, comparison techniques can sometimes be used by noting that if $0 < f_1(n) \leqslant f_2(n)$ for $n \geqslant k$ and E_n^i is the error term for f_i $(i = 1,2)$, then $E_n^1 < E_n^2$ for $n \geqslant k$. Our next example illustrates this.

EXAMPLE 4. The sum of the first 41 terms, S_{41}, approximates $\Sigma_{n=2}^{\infty}(n^3 + 5n - 7)^{-4/3}$ accurately to at least 5 decimal places. To see this, note that $(n^3 + 5n - 7)^{-4/3} < n^{-4}$ for $n \geqslant 2$. Thus, if E_n denotes the error term for $\Sigma_{n=2}^{\infty} n^{-4}$, then by Example 2, $E_m < .5(10)^{-5}$ if $m^{-3}/3 < .5(10)^{-5}$ or $m \geqslant 41$. We thereby obtain 5-place accuracy for the original series.

None of the preceding—examples or theory—requires the concept of the definite integral. However, if the definite and improper integrals have been introduced, the Integral Test is an immediate corollary to Theorem (A). Since $\ln x$ and e^x would then be available, we could consider examples of the following type.

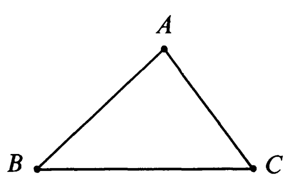
EXAMPLE 5. Inequality (1) assures us that S_4 approximates $\Sigma_{n=1}^{\infty} e^{-n^2} n$ accurately to 4 places, since if $g(x) = -e^{-x^2}/2$, $-g(m) < .5(10)^{-4}$ when $m \geqslant 4 > 2\sqrt{\ln 10}$.

Reference

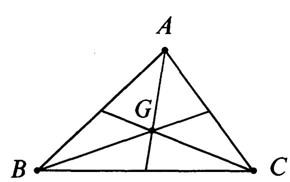
[1] Howard Anton, Calculus with Analytical Geometry, Wiley, New York, 1980.

Triangle Rhyme

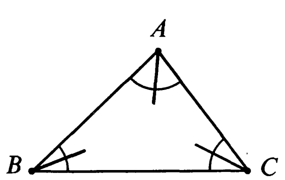
DWIGHT PAINE
Messiah College
Grantham, PA



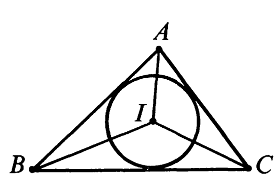
Let's talk about a **triangle**;
We'll call it ABC,



With **medians** converging
To the centroid, labeled G.



To bisect all the angles
Takes a dreadful steady eye,



But all the **angle bisectors**
Meet at the **incenter** I.