Polynomial Translation Groups

DAN KALMAN
University of Wisconsin
Green Bay, WI 54302

Consider the 4×4 matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r^2 & 2r & 1 & 0 \\ r^3 & 3r^2 & 3r & 1 \end{bmatrix}, \ r \text{ a scalar.}$$

Is this set of matrices closed under multiplication? Inversion? Do two such matrices commute? (You have probably answered "yes" to each question on the basis of gamesmanship. If so, it is probable that you also view these conclusions as unexpected or striking. Hence, it would have been more to the point to begin "Isn't it interesting that this set is closed under multiplication," etc. But then I couldn't have written ...). Yes, yes, and yes. Let me show you a reason why.

If $i \leqslant j$, the ij entry of the matrix above is given by

$$m_{ij}(r) = \binom{i-1}{j-1} r^{i-j}, \tag{1}$$

that is, the ith row contains the terms of the binomial expansion of $(r+1)^{i-1}$. As there is no particular virtue in considering 4×4 matrices, let n be a fixed positive integer, and define for each number r a lower triangular $n \times n$ matrix $M(r)$ according to (1). Then

$$M(r)M(s) = M(r+s), \tag{2}$$

so that the set of matrices $M(r)$ is a multiplicative group.

Arm yourself from your favorite arsenal of combinatorial identities and you are sure to emerge from battle with (2) the victor. However, once I convince you that geometrically $M(r)$ is a translation of r units, you will perceive the validity of (2) without recourse to such weaponry. On the contrary, (2) will yield up an identity or two to add to the arsenal.

Every translation requires two things: something to move and a space to move it in. The translations of interest here move polynomials about in the Cartesian plane. Consider the space of polynomials $p(x)$ of degree less than n, and identify each polynomial with its graph. Each polynomial may be represented by a vector of coefficients with respect to the basis $\mathbf{B} = \{1, x, x^2, \dots, x^{n-1}\}$, and linear transformations can then be represented by matrices. In particular, let the transformation T_r move a polynomial r units to the left. That is, if p is a polynomial, $T_r p$ is the polynomial defined by

$$T_r p(x) = p(x+r). \tag{3}$$

By evaluating T_r at each element of $\mathbf{B}$, and expressing the results as coefficient vectors, we derive the rows of $M(r)$. Thus, if the coefficient vector of p is $\mathbf{v}$ and that of $T_r p$ is $\mathbf{w}$, then $\mathbf{w} = \mathbf{v}M(r)$; $M(r)$ represents a translation of r units to the left. And, since composing translations of r units and s units yields a translation of $r+s$ units, (2) is established. Moreover, by (2), the mapping $r \to M(r)$ establishes an isomorphism from the additive group of the real numbers into the multiplicative structure of the $n \times n$ matrices. The image of this mapping is consequently a commutative group. This result applies more generally to polynomials and matrices over an arbitrary field if translations are defined as in (3).

Having established (2) by exploiting a fortuitous geometric interpretation, it is rewarding to consider the toil avoided. To derive (2) directly, it is necessary to prove

$$\sum_{k=1}^{n} m_{ik}(r) m_{kj}(s) = m_{ij}(r+s),$$

which, after using (1) and omitting the zero terms from the sum, becomes

$$\sum_{k=j}^{i} \binom{i-1}{k-1} r^{i-k} \binom{k-1}{j-1} s^{k-j} = \binom{i-1}{j-1} (r+s)^{i-j};\ 1 \leqslant j \leqslant i \leqslant n. \tag{4}$$

Luckily, we already know (2) is true and so we get (4) gratis, avoiding the effort of a direct derivation. Moreover, other identities can be deduced easily from (4). For example, replacing i and j by $I+1$ and $J+1$, respectively, eliminating the irrelevant reference to n, and reindexing the sum to run from J to I gives

$$\sum_{k=J}^{I} \binom{I}{k}\binom{k}{J} r^{I-k} s^{k-J} = \binom{I}{J}(r+s)^{I-J};\ 0 \leqslant J \leqslant I. \tag{5}$$

This identity may be viewed as a generalization of the binomial theorem (to which it reduces when $J=0$). As special cases, take $r=1$ and $s=1$ or -1 to derive

$$\binom{I}{J} 2^{I-J} = \sum_{k=J}^{I} \binom{I}{k}\binom{k}{J} \text{ and } \sum_{k=J}^{I} (-1)^k \binom{I}{k}\binom{k}{J} = 0.$$

The second of these identities appears as entry 3.119 in [**1**]. Similarly, when $r=1$, identity (5) is essentially the same as entry 3.118 of [**1**], which is a normalized version of (5).

We have shown that equation (5) has interpretations in terms of geometry and matrix algebra. There is also a combinatorial interpretation. Imagine a painter who is to paint strips of paper using $r+s$ colors, in the following way. Each strip is divided into I sections. Of these, J are to remain blank, the remaining $I-J$ must be painted using any color of the $r+s$, one color for each section. The painter may produce, in this fashion, $\binom{I}{J}(r+s)^{I-J}$ differently painted strips. Alternatively, let the painter first choose some number k of the I sections (with k no less than J) and paint the remaining $I-k$ sections using the first r colors. Now k sections are blank. The painter selects J of these to leave blank, and paints the other $k-J$ using s colors. Following these steps, there are

$$\sum_{k=J}^{I} \binom{I}{k} r^{I-k} \binom{k}{J} s^{k-J}$$

different ways to paint a strip. In following either set of instructions, the artist can produce any painted strip with J blank parts, so the formulas enumerating the set of outcomes must agree, which verifies (5).

Buried in (5) somewhere is the combinatorial equivalent of $M(r)M(-r)=I$. The geometric justness of this result is evident but the explicit matrix formulation is a pleasant surprise. To invert $M(r)$, simply change the sign of every other entry. For example, the inverse of the 4×4 matrix displayed at the beginning of this note is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -r & 1 & 0 & 0 \\ r^2 & -2r & 1 & 0 \\ -r^3 & 3r^2 & -3r & 1 \end{bmatrix}.$$

As a final topic, I'd like to point out some ways to exploit this example. Note first that replacing r with a numerical value disguises the pattern of the matrix. Thus, given the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 4 & 1 & 0 \\ 8 & 12 & 6 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 9 & 6 & 1 & 0 \\ 27 & 27 & 9 & 1 \end{bmatrix}$$

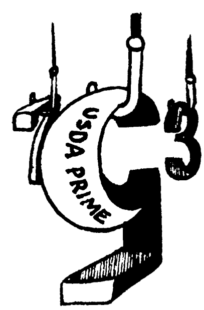
as $M(2)$, $M(-1)$, and $M(3)$, respectively, a student might find it challenging to determine the general form, $M(r)$. In a similar vein, as a source of matrices with known inverses, the translation matrices are useful for concocting matrix inversion exercises. Students may find a certain amount of charm in setting out to invert a 4×4 matrix and ending up (nearly) back at the original matrix.

More abstract problems can be designed to exercise a student's understanding of axioms. For instance, having a student check the set of 3×3 $M(r)$'s for closure under multiplication provides practice with axioms, and in particular, that troublesome notion of closure. Posing similar problems without specifying n adds another increment in difficulty. Finally, the example may be presented much as it is here. The specific lesson that an appropriate geometric interpretation reveals algebraic structure and combinatorial identities is a valuable one. More generally, this example may hint at the fertility that results from multiple interpretations of a single mathematical structure through the use of isomorphisms.

Reference

[1] Henry W. Gould, Combinatorial Identities, Morgantown Printing and Binding, Morgantown, West Virginia, 1972.

In his *prime*



USDA *prime*

The *prime* coat

***Prime* observations 2.**