Variations of the Sliding Ladder Problem

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Calculus teachers are familiar with the sliding ladder problem from the study of related rates. However, it is not well known that some versions of this problem are physically impossible. For example, consider the version in which a ladder is leaning against a wall while the bottom is pulled away at a constant velocity. Most often, students are asked to find the velocity of the top of the ladder at a given instant. This is a straightforward problem and there are no surprising results. In some books however the students are asked to consider what happens when the ladder hits the floor. The surprising result in this case is that the velocity of the top approaches infinity.

One can easily be convinced by performing simple experiments using a pen or a ruler that sliding objects like the ladder never achieve very high speeds. A simple energy argument can also be used to show that a sliding ladder on its own will not attain very high velocities. A ladder moving at velocities approaching infinity contains an amount of energy that also approaches infinity. On the other hand, a ladder leaning against a wall contains only a moderate amount of energy in the form of potential energy $V = mgh$, where m is the mass of the ladder, g is the gravitational acceleration, and h is the height of the center of mass above the floor. It is this energy that will be converted to kinetic energy if the ladder begins to slide along the wall and the floor under the influence of its own weight. In the real world even some of this energy will be converted to thermal energy by friction and will not be converted to kinetic energy. The immediate conclusion is simply that there isn't enough energy available for the ladder to achieve very high speeds. We note that there is no error in the mathematics used to solve the ladder problem stated above. How then can this apparent contradiction be resolved? To answer this question we must consider the physics involved in the problem.

In this note we analyze the sliding ladder problem using elementary notions of mechanics. The motion of the ladder is governed by the equations for rigid body motion. These are $\mathbf{F} = m\mathbf{a}_G$, and $M_G = I_G\ddot{\theta}$, where $\mathbf{F}$ is the vector sum of the external forces on the ladder, m is the mass of the ladder, $\mathbf{a}_G$ is the acceleration of the center of mass G, M_G is the total torque of the external forces about G, I_G is the moment of inertia of the ladder about G, and $\ddot{\theta}$ is the angular acceleration of the ladder. The first equation determines the acceleration of the center of mass, the second determines the angular acceleration of the ladder. Furthermore, when gravity is the only force that does work we also use the equation of conservation of energy $T + V =$ constant, where $T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\dot{\theta}^2$ is the kinetic energy, and $V = mgh$ is the gravitational potential energy. For simplicity we model the ladder by a uniform rod of length $2l$, so that $I_G = \frac{1}{3}ml^2$.

We consider three variations of the sliding ladder problem, which in each case, and taken together, provide interesting insights into the ladder problem and resolve the paradox of infinite speed.

Unconstrained ladder

In the first problem a ladder is leaning against a wall and sliding under the influence of gravity alone. The formal statement of this problem is as follows.

Problem 1. A uniform rod of mass m and length $2l$ is leaning against a wall at an angle α. It is released from rest and slides in the xy-plane along a smooth wall and smooth floor. Determine the motion of the ladder and the speed of the top as the ladder hits the floor.

It turns out that the rod leaves the wall when the top has fallen one third of the way down the wall. After this instant, the equations $x_B^2 + y_A^2 = 4l^2$ and $x_B\dot{x}_B + y_A\dot{y}_A = 0$ no longer hold and therefore we cannot conclude that $\dot{y}_A \to \infty$ as $y_A \to 0$.

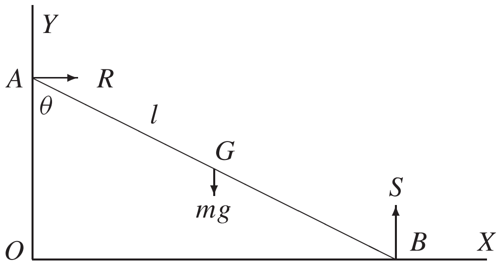

Figure 1. Motion of a sliding ladder.

Let $\theta = \theta(t)$ denote the angle at the top of the rod at time t, let R and S denote the normal reactions at the top A and bottom B respectively, and mg the weight of the rod acting at the center of mass $G(x_G, y_G)$. While there is contact with the wall,

$$x_G = l\sin\theta \quad \text{and} \quad y_G = l\cos\theta.$$

Thus $\mathbf{v}_G = \dot{x}_G\mathbf{i} + \dot{y}_G\mathbf{j} = (l\cos\theta\,\dot{\theta})\,\mathbf{i} - (l\sin\theta\,\dot{\theta})\,\mathbf{j}$, so that $v_G = l\dot{\theta}$. Hence the kinetic energy of the rod at position θ is

$$\begin{aligned} T &= \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\dot{\theta}^2 \\ &= \tfrac{1}{2}ml^2\dot{\theta}^2 + \tfrac{1}{6}ml^2\dot{\theta}^2 = \tfrac{2}{3}ml^2\dot{\theta}^2. \end{aligned}$$

Initially, the kinetic energy is zero. The potential energy at position θ is $mgl\cos\theta$, whereas the initial potential energy is $mgl\cos\alpha$. Since the contact forces R and S do no work, energy is conserved, and so

$$\tfrac{2}{3}ml^2\dot{\theta}^2 + mgl\cos\theta = mgl\cos\alpha,$$

from which

$$\dot{\theta}^2 = \frac{3g}{2l}(\cos\alpha - \cos\theta).$$

Differentiating with respect to t we obtain

$$\ddot{\theta} = \frac{3g}{4l}\sin\theta.$$

Next, taking the x-component of the equation $\mathbf{F} = m\mathbf{a}_G$, we obtain

$$\begin{aligned} R = m\ddot{x}_G &= ml(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2) \\ &= \tfrac{3}{4}mg\sin\theta\,(3\cos\theta - 2\cos\alpha). \end{aligned}$$

Hence

$$R = 0 \quad \text{when} \quad \cos\theta = \tfrac{2}{3}\cos\alpha.$$

That is, the upper end of the rod loses contact with the wall when it has fallen one third of its original height.

After losing contact, the rod continues to move away from the wall, and the motion is determined only by gravity and the reaction S of the floor. An analysis of this motion shows that when the rod hits the floor the velocity of the top is

$$\sqrt{\tfrac{2}{9}gl\cos^3\alpha}\,\mathbf{i} - \sqrt{\tfrac{2}{3}gl(9\cos\alpha - \cos^3\alpha)}\,\mathbf{j},$$

so that the speed is $\sqrt{gl(6\cos\alpha - \tfrac{4}{9}\cos^3\alpha)}$. Note that for a given initial angle this speed is proportional to $\sqrt{l}$.

Constrained ladder

We now turn to what happens if the ladder is not permitted to leave the wall, and therefore the equation $x_B^2 + y_A^2 = 4l^2$ must hold. The formal statement of this version is as follows.

Problem 2. The top of a uniform rod of mass m and length $2l$ is attached by a smooth ring to the y-axis, at an angle α. It is released from rest and slides in the xy-plane along a smooth floor. Determine the motion of the rod and the speed of the top as the rod hits the floor.

It turns out that when the rod hits the floor the speed $\dot{x}_B$ of the bottom is zero. Therefore the equation $x_B\dot{x}_B + y_A\dot{y}_A = 0$ does not determine $\dot{y}_A$, and in particular, we cannot conclude that $\dot{y}_A \to \infty$ as $y_A \to 0$.

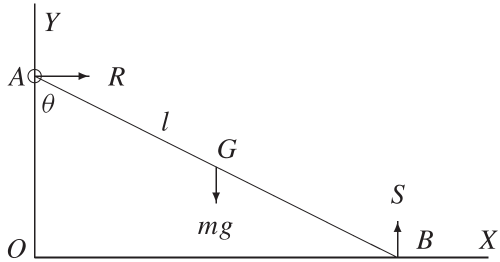

Figure 2. Ladder maintains contact with wall.

As before, we have

$$\dot{\theta}^2 = \frac{3g}{2l}(\cos\alpha - \cos\theta).$$

For $\alpha \leq \theta \leq \pi/2$, the position vectors of the top A and bottom B are $\mathbf{r}_A = (2l\cos\theta)\,\mathbf{j}$ and $\mathbf{r}_B = (2l\sin\theta)\,\mathbf{i}$. Differentiating we find that the velocities of the ends are $\mathbf{v}_A = (-2l\sin\theta\,\dot{\theta})\,\mathbf{j}$ and $\mathbf{v}_B = (2l\cos\theta\,\dot{\theta})\,\mathbf{i}$, so that the speeds are

$$v_A = 2l\sin\theta\,\dot{\theta} = \sin\theta\sqrt{6gl(\cos\alpha - \cos\theta)} \quad \text{and}$$
$$v_B = 2l\cos\theta\,\dot{\theta} = \cos\theta\sqrt{6gl(\cos\alpha - \cos\theta)}.$$

These equations reveal several interesting facts about the motion of the rod. Starting with v_B, we find that $dv_B/d\theta$ is initially positive, it is zero at $\theta = \cos^{-1}(\frac{2}{3}\cos\alpha)$, and then it becomes negative. Thus, v_B is greatest at $\theta = \cos^{-1}(\frac{2}{3}\cos\alpha)$, which is the same height at which the rod leaves the wall in the previous problem. This greatest speed is $\frac{2}{3}\sqrt{2gl\cos^3\alpha}$. We also see that as the rod hits the floor, $v_B = 0$. Thus v_B increases from 0 to $\frac{2}{3}\sqrt{2gl\cos^3\alpha}$, then decreases to 0. As we noted earlier, because of this result, the equation $x_B\dot{x}_B + y_A\dot{y}_A = 0$ does not determine $\dot{y}_A$ when $y_A = 0$. However, the equation for v_A does give, for the speed of the top of the rod as it hits the floor, the finite value $\dot{y}_A = v_A = \sqrt{6gl\cos\alpha}$. We note that $dv_A/d\theta > 0$, so that v_A increases monotonically from 0 to $\sqrt{6gl\cos\alpha}$. For example, when $\alpha = 30°$ and $2l = 4\,\text{m}$, the final velocity of the top is $10.1\,\text{ms}^{-1}$ (approximately).

We can also use $dv_B/d\theta$ to determine the variation of the reaction R of the wall. Taking the x-component of the equation $\mathbf{F} = m\mathbf{a}_G$ gives

$$R = m\ddot{x}_G = \frac{1}{2}m\ddot{x}_B = \frac{1}{2}m\frac{dv_B}{dt} = \frac{1}{2}m\frac{dv_B}{d\theta}\frac{d\theta}{dt}.$$

Since $d\theta/dt$ is positive, we see that R is initially positive, it is zero at $\theta = \cos^{-1}(\frac{2}{3}\cos\alpha)$, and then it becomes negative in order to keep the rod in contact with the wall.

Constrained ladder with constant velocity

Finally we consider a ladder where the bottom is moving at a constant speed v_0, while the ladder maintains contact with the wall. In the previous two problems we found that when the ladder is moving under the influence of gravity alone, then the bottom of the ladder does not move at constant speed. Therefore, in order to maintain a constant speed of the bottom while the ladder stays in contact with the wall, additional forces on the ladder are required. We find that this motion can be realized by pulling down at the top and pushing back at the bottom. Here is a formal statement of this problem.

Problem 3. A uniform rod of mass m and length $2l$ slides along a smooth wall and smooth floor. Determine the forces that need to be applied at the top and bottom of the rod so that it maintains contact with the wall while the bottom moves at a constant speed v_0.

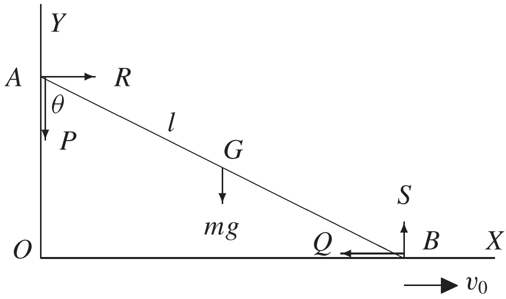

Figure 3. *B* moves with constant velocity.

For the endpoint B we find

$$x_B = 2l\sin\theta \quad \text{and} \quad \dot{x}_B = 2l\cos\theta\,\dot{\theta} = v_0.$$

Hence

$$\dot{\theta} = \frac{v_0}{2l\cos\theta} \quad \text{and} \quad \ddot{\theta} = \frac{v_0^2\sin\theta}{4l^2\cos^3\theta}.$$

For G we find

$$x_G = l\sin\theta, \qquad y_G = l\cos\theta,$$

$$\dot{x}_G = l\cos\theta\,\dot{\theta} = \frac{1}{2}v_0, \qquad \dot{y}_G = -l\sin\theta\,\dot{\theta} = -\frac{1}{2}v_0\tan\theta,$$

$$\ddot{x}_G = 0, \qquad \ddot{y}_G = -\frac{1}{2}v_0\sec^2\theta\,\dot{\theta} = -\frac{v_0^2}{4l\cos^3\theta}.$$

The x- and y-components of the equation $\mathbf{F} = m\mathbf{a}_G$ are

$$R - Q = m\ddot{x}_G \quad \text{and} \quad S - P - mg = m\ddot{y}_G$$

respectively; hence

$$R = Q \quad \text{and} \quad S = P + mg - \frac{mv_0^2}{4l\cos^3\theta}.$$

The equation for rotational motion $M_G = I_G\ddot{\theta}$ becomes

$$Pl\sin\theta + Sl\sin\theta - 2Rl\cos\theta = \frac{1}{3}ml^2\ddot{\theta} = \frac{mv_0^2\sin\theta}{12\cos^3\theta}.$$

We deduce that

$$R = \left(P + \frac{1}{2}mg - \frac{mv_0^2}{6l\cos^3\theta}\right)\tan\theta.$$

In order that the rod maintains contact with the wall and floor, both R and S must be non-negative, so that

$$P \geq \max\left\{\frac{mv_0^2}{6l\cos^3\theta} - \frac{1}{2}mg,\ \frac{mv_0^2}{4l\cos^3\theta} - mg\right\}.$$

If we let

$$E = \frac{mv_0^2}{6l\cos^3\theta} - \frac{1}{2}mg \quad \text{and} \quad F = \frac{mv_0^2}{4l\cos^3\theta} - mg,$$

then

$$\max\{E, F\} = \begin{cases} E & \text{if } \cos^3\theta \geq v_0^2/6gl, \\ F & \text{if } \cos^3\theta \leq v_0^2/6gl. \end{cases}$$

Thus we can take $P = \max\{E, F\}$, and then set $Q = R$. As $\theta \to \pi/2$, both P and Q tend to infinity.

We remark that this situation does yield arbitrarily large speeds. Achieving this motion requires the application of constraint forces that approach infinity. The conclusion is that this motion is unrealistic.

Conclusions

In terms of mechanics, the calculus related rates problem is a problem in kinematics, in that there is no regard for the forces that cause the motion. These three examples illustrate that it is only when we consider the forces that cause the motion that we can resolve the paradox of speeds approaching infinity. They also indicate caution in regarding the sliding ladder as an example of a real world application of mathematics.

It is well known that a sliding ladder, left to fall under the action of gravity alone, leaves the wall when the upper end has fallen one third of its original height; for instance, see Cahn and Nadgorny [**1**]. Moreover, the mechanics of a physical ladder in which the lower end is kept at a constant speed is addressed by Freeman and Palffy-Muhoray [**2**] and Scholten and Simoson [**3**]. However, our account of the ladder problem is different, and moreover presents a synthesis of several perspectives on the sliding ladder paradox.

References

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