Coin Flipping, Dynamical Systems, and the Fibonacci Numbers

STEPHEN F. KENNEDY[1]
Saint Olaf College
Northfield, MN 55057

MATTHEW W. STAFFORD[2]
University of Texas
Austin, TX 78712

In this note we pose a pair of seemingly disparate problems, one from probability and the other from dynamical systems. We will show that these two problems are actually one—i.e., their solutions are identical and each, surprisingly, involves the Fibonacci numbers. The dynamical systems problem is new. Variants of the probability problem have been around for centuries. The oldest reference we have found is Problem LXXIV in Abraham De Moivre's *The Doctrine of Chances* [**3**], the first edition of which appeared in 1718. (See [**6**] for a treatment of De Moivre's formulation with a more modern flavor.) After solving both problems and finding the connection between them, we will mention some simple generalizations.

Suppose we are asked to flip a fair coin repeatedly until two consecutive heads appear. What is the probability that this will ever occur? And, more specifically: What is the probability that it will occur in exactly n tosses? The dynamical systems problem arose during the second author's dissertation research [**1**, **5**]. If $f(x) = 2x \pmod 1$, for $x \in [0, 1)$, where are all the points that visit $[3/4, 1)$ for the first time after exactly n iterations of $f(x)$? That is, let $A_n = \{x \mid f^n(x) \in [3/4, 1)$, and n is minimal with respect to this property$\}$. What is the geometry of A_n? Is it the case that $\bigcup_{n=0}^{\infty} A_n = [0, 1)$? We note that $f^n(x)$ denotes the composition $f \circ f \circ f \circ \cdots \circ f(x)$, of n copies of f.

A moment's thought should convince you that the probability of seeing two heads in a row in the coin flipping problem is very large. Is it one? Represent a sequence of tosses by a sequence of H's and T's in the obvious way. Call a sequence *n-allowable* if it ends the coin tossing after n flips, i.e., has 'HH' as its last two letters and no earlier occurrence of this string. Since all sequences are equally probable, the probability of stopping after n tosses is the number of n-allowable sequences (call this a_n) divided by 2^n. To count the n-allowable sequences notice that every n-allowable sequence is of one of two forms: a 'T' followed by an $(n-1)$-allowable sequence, or, an 'HT' followed by an $(n-2)$-allowable sequence. Moreover, appending 'T' to the beginning of any $(n-1)$-allowable sequence yields an n-allowable one, as does appending 'HT' to any $(n-2)$-allowable sequence. Therefore, $a_n = a_{n-1} + a_{n-2}$ and $a_2 = a_3 = 1$. That is, a_n is the $(n-2)^{\text{nd}}$ Fibonacci number. The probability of stopping after n tosses is then $a_n/2^n$, and the probability of *ever* stopping is $\sum_{n=2}^{\infty} a_n/2^n$. We expect that this sum must converge to one: We'll prove this momentarily.

To solve the dynamical systems problem, it pays to work in dyadic expansions. If $.101101\ldots$ is the dyadic expansion of x, then $.01101\ldots$ is the dyadic expansion of

[1]Much of the work in this paper was performed while the author was a postdoctoral visitor at the University of Delaware.

[2]The author acknowledges the support of The Institute for Mathematics and Its Applications, University of Minnesota.

$f(x)$ ($f(x)$ "forgets" the first digit of x's dyadic expansion). Any number in $[3/4, 1)$ has one as the first two digits of its dyadic expansion, therefore a number that visits this interval for the first time after n iterations of f must have the string '11' in its $(n+1)^{\text{st}}$ and $(n+2)^{\text{nd}}$ places and no earlier occurrences of this string. The number of different $(n+2)$-length strings that could begin the expansion of a point in A_n is equal to the number of $(n+2)$-allowable words in the coin flipping problem. The collection of points that have a specified $(n+2)$-length string at the beginning of their expansion is an interval of length $1/2^{n+2}$. Therefore, A_n is a collection of a_{n+2} disjoint (why?) intervals each of this length. If $n \neq m$, then $A_n \cap A_m = \varnothing$ and the length of $\bigcup_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} a_{n+2}/2^{n+2}$, a reindexing of the earlier sum.

As we shall see, this converges to one, however, not every point eventually maps into $[3/4, 1)$. For example, $1/3 = .010101\ldots$, $f(1/3) = .101010\ldots = 2/3$, and $f(2/3) = 1/3$. In some sense these points correspond to coin flipping sequences that never terminate (e.g., HTTHTT...), but we have some extra structure in this situation. The set $[0, 1)$ inherits a topology as a subset of the real line and we have a function, $f(x)$, which is *invariant* on the set of points that never visit $[3/4, 1)$. (Invariant means if x is a point in this set, then $f(x)$ is also.) In this topology our set is a Cantor set and together with $f(x)$ forms a chaotic dynamical system [4].

To prove that $\sum_{n=2}^{\infty} a_n/2^n = 1$, first notice that by the ratio test the sum does converge. Let

$$\begin{aligned} S &= \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{2}{16} + \tfrac{3}{32} + \tfrac{5}{64} + \cdots \\ \tfrac{1}{2}S &= \qquad \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{2}{32} + \tfrac{3}{64} + \cdots \\ \tfrac{3}{2}S &= \tfrac{1}{4} + \tfrac{2}{8} + \tfrac{3}{16} + \tfrac{5}{32} + \tfrac{8}{64} + \cdots \\ &= 2S - \tfrac{1}{2} \\ \Rightarrow S &= 1. \end{aligned}$$

This is quite remarkable—the sum whose i^{th} term is the i^{th} Fibonacci divided by 2^{i+2} converges to one! That we can even find the number to which this series converges is unusual; that its sum is the simplest rational seems miraculous. The explanation for this miracle lies in the so-called "generating function" for the Fibonacci numbers. Let $g(x)$ be the function whose power series expansion has n^{th} term the n^{th} Fibonacci number times x^n. It can be shown relatively easily that [2]:

$$g(x) = \sum_{n=0}^{\infty} a_{n+2} x^n = \frac{1}{1 - x - x^2}.$$

Our series is $(1/4)g(1/2) = 1$.

De Moivre [3] asked for the probability of observing three consecutive heads in a sequence of 10 flips. More generally, what is the probability of seeing m consecutive heads in *exactly n* flips? For three consecutive heads the numbers of n-allowable words are $1, 1, 2, 4, 7, 13, \ldots$. The recursion relation is $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. The proof is a slight modification of the original: We can append a 'T' onto any $(n-1)$-allowable word, an 'HT' onto any $(n-2)$-allowable word, or an 'HHT' onto an $(n-3)$-allowable word. If we form the power series whose n^{th} term is a_{n-3} times x^n, we get the generating function $g(x) = 1/(1 - x - x^2 - x^3)$. Notice that $(1/8)g(1/2) = 1$, the probability of ever seeing three heads is one. Table 1 lists the first few terms, the recursion relation, and the resulting generating function for $m = 1, 2, 3, 4, 5$. The patterns persist.

In solving these problems we have also solved the corresponding dynamical systems problems: What is the set of points that visits $[1 - 1/2^m, 1)$ after n iterations of $f(x)$? If we ask what happens if we have a p-sided coin and we want two consecutive heads, the results are summarized in Table 2. Again, it is not difficult to prove that the patterns persist.

TABLE 1

m	Terms	Recursion	Generating Function
1	$1, 1, 1, 1, 1, \ldots$	$a_n = a_{n-1}$	$g(x) = \frac{1}{1-x}$
2	$1, 1, 2, 3, 5, 8, \ldots$	$a_n = a_{n-1} + a_{n-2}$	$g(x) = \frac{1}{1-x-x^2}$
3	$1, 1, 2, 4, 7, 13, \ldots$	$a_n = a_{n-1} + a_{n-2} + a_{n-3}$	$g(x) = \frac{1}{1-x-x^2-x^3}$
4	$1, 1, 2, 4, 8, 15, 29, \ldots$	$a_n = a_{\ -1} + \cdots + a_{n-4}$	$g(x) = \frac{1}{1-x-x^2-x^3-x^4}$
5	$1, 1, 2, 4, 8, 16, 31, \ldots$	$a_n = a_{n-1} + \cdots + a_{n-5}$	$g(x) = \frac{1}{1-x-x^2-x^3-x^4-x^5}$

TABLE 2

p	Terms	Recursion	Generating Function
2	$1, 1, 2, 3, 5, 8, \ldots$	$a_n = a_{n-1} + a_{n-2}$	$g(x) = \frac{1}{1-x-x^2}$
3	$1, 2, 6, 16, 44, \ldots$	$a_n = 2(a_{n-1} + a_{n-2})$	$g(x) = \frac{1}{1-2x-2x^2}$
4	$1, 3, 12, 45, 171, \ldots$	$a_n = 3(a_{n-1} + a_{n-2})$	$g(x) = \frac{1}{1-3x-3x^2}$
5	$1, 4, 20, 96, 464, \ldots$	$a_n = 4(a_{n-1} + a_{n-2})$	$g(x) = \frac{1}{1-4x-4x^2}$

The reader is urged to consider the problem of obtaining m consecutive heads from a p-sided coin. Also, the numbers obtained by finding the limits of quotients of consecutive terms for the various values of p have many nice properties, analogous to properties of the golden mean ($p = 2$).

Acknowledgement. The authors would like to thank the many people whose advice and suggestions contributed to this note: R. O. Young, W. T. Butterworth, R. Holmgren, and R. F. Williams who started it all.

REFERENCES

1. J. Ashley, B. Kitchens, and M. Stafford, Boundary sets of Markov partitions, *Trans. AMS*, (333), Sept. 1992, pp. 177–201.
2. G. Bergman and K. D. Fryer, *Introduction to Combinatorics*, Academic Press, London, 1972.
3. A. De Moivre, *The Doctrine of Chances*, 3rd edition, 1756, Chelsea Publishing Co., New York (reprint), 1967.
4. R. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd edition, Addison-Wesley Publishing Co., Reading, MA, 1989.
5. M. Stafford, Markov partitions for the doubling map, Ph.D. Thesis, Northwestern University, Evanston, IL, 1989.
6. I. Todhunter, *A History of the Mathematical Theory of Probability*, 1865, Chelsea Publishing Co., New York (reprint) 1965.