Sines and Cosines of Angles in Arithmetic Progression

MICHAEL P. KNAPP
Loyola University Maryland
Baltimore, MD 21210-2699
mpknapp@loyola.edu

doi:10.4169/002557009X478436

In a recent Math Bite in this MAGAZINE [**2**], Judy Holdener gives a physical argument for the relations

$$\sum_{k=1}^{N} \cos\left(\frac{2\pi k}{N}\right) = 0 \qquad \text{and} \qquad \sum_{k=1}^{N} \sin\left(\frac{2\pi k}{N}\right) = 0,$$

and comments that "It seems that one must enter the realm of complex numbers to prove this result." In fact, these relations follow from more general formulas, which can be proved without using complex numbers. We state these formulas as a theorem.

THEOREM. *If $a, d \in \mathbb{R}$, $d \neq 0$, and n is a positive integer, then*

$$\sum_{k=0}^{n-1} \sin(a+kd) = \frac{\sin(nd/2)}{\sin(d/2)} \sin\left(a + \frac{(n-1)d}{2}\right)$$

and

$$\sum_{k=0}^{n-1} \cos(a+kd) = \frac{\sin(nd/2)}{\sin(d/2)} \cos\left(a + \frac{(n-1)d}{2}\right).$$

We first encountered these formulas, and also the proof given below, in the journal *Arbelos*, edited (and we believe almost entirely written) by Samuel Greitzer. This journal was intended to be read by talented high school students, and was published from 1982 to 1987. It appears to be somewhat difficult to obtain copies of this journal, as only a small fraction of libraries seem to hold them.

Before proving the theorem, we point out that, though interesting in isolation, these formulas are more than mere curiosities. For example, the function $D_m(t) = \frac{1}{2} + \sum_{k=1}^{m} \cos(kt)$ is well known in the study of Fourier series [**3**] as the Dirichlet kernel. This function is used in the proof of Dirichlet's theorem: If a function $f(t)$ is continuous on $[-\pi, \pi]$ and has $f(-\pi) = f(\pi)$, then the Fourier series for $f(t)$ converges to $f(t)$ at every point of $[-\pi, \pi]$. In fact, Pinkus and Zafrany [**3**] prove that

$$D_m(t) = \frac{\sin\left(m + \frac{t}{2}\right)}{2\sin\left(\frac{t}{2}\right)}$$

using the same method as Greitzer.

Proof of the Theorem. We reiterate that this proof is not original with the author. Greitzer [**1**] proves the first formula and leaves the second as an exercise for the reader. Therefore, in this note we will prove the second formula and refer the reader to Grietzer for the other, which proceeds along exactly the same lines. First, recall the trigonometric identity

$$\sin(A+B)-\sin(A-B)=2\cos(A)\sin(B). \tag{1}$$

Now, if we write C for the sum we wish to evaluate, then multiplying by $2\sin(d/2)$ and using (1) yields

$$\begin{aligned}2C\cdot\sin(d/2) &= \sum_{k=0}^{n-1} 2\cos(a+kd)\sin(d/2)\\ &= \sum_{k=0}^{n-1} \sin\left(a+\frac{(2k+1)d}{2}\right)-\sin\left(a+\frac{(2k-1)d}{2}\right)\\ &= \sin\left(a+\frac{(2n-1)d}{2}\right)-\sin\left(a-\frac{d}{2}\right)\\ &= 2\cos\left(a+\frac{(n-1)d}{2}\right)\sin(nd/2),\end{aligned}$$

where the third equality follows because the series telescopes. Solving our final equation for C completes the proof of the theorem. ■

REFERENCES

1. S. Greitzer, Many cheerful facts, *Arbelos* **4**:5 (1986) 14–17.
2. J. Holdener, Math bite: sums of sines and cosines, this MAGAZINE, **82**:2 (2009) 126.
3. Allan Pinkus and Samy Zafrany, *Fourier Series and Integral Transforms*, Cambridge University Press, 1997.

Black Holes through *The Mirrour*

ANDREW J. SIMOSON
King College, Bristol, TN 37620
ajsimoso@king.edu

doi:10.4169/193009809X468724

What is the path of a free-falling stone when dropped from the surface of a spinning earth? This riddle is an old one. By the late Middle Ages it was common knowledge. In the first printed book with illustrations in the English language, William Caxton in a popular encyclopedia of what everyone should know, called *The Mirrour of the World*, which appeared in 1481 and was a translation of a French manuscript of 1245, which in turn was a translation of an earlier text in Latin, said [**1**, p. 55],

> And if the earth were pierced through in two places, of which that one hole were cut into the other like a cross, and four men stood right at the four heads of these two holes, one above and another beneath, and like-wise on both sides, and that each of them threw a stone into the hole, whether it were great or little, each stone should come into the middle of the earth without ever to be removed from thence.